Banach-Mazur games played with arrows

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- Let **V** be the family of all G_δ subsets of X.

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The result of a play is the co-limit

$$K_\infty = \lim \{K_n\}_{n \in \omega} \in \mathrm{Obj}(\mathbf{V}).$$

Definition

Let $\mathscr{W}$ be a class of $\mathbf{V}$-objects. We say that Odd wins if he has a strategy such that no matter how Eve plays, the co-limit of the resulting sequence is isomorphic to some element of $\mathscr{W}$.
Denote this game by $\mathrm{BM}(\mathfrak{K}, \mathscr{W})$.

Theorem

Assume Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, \mathscr{W}_n)$*, where each* $\mathscr{W}_n$ *is closed under isomorphisms. Then Odd has a winning strategy in*

$$\mathrm{BM}(\mathfrak{K}, \bigcap_{n\in\omega} \mathscr{W}_n).$$

Definition

Let $\mathfrak{S}$ be another category and let $\Phi\colon \mathfrak{S} \to \mathfrak{K}$ be a covariant functor. We say that Φ is dominating if

(D1) For every $X \in \operatorname{Obj}(\mathfrak{K})$ there is $s \in \operatorname{Obj}(\mathfrak{S})$ such that $\mathfrak{K}(X, \Phi(s)) \neq \emptyset$.

(D2) Given $s \in \operatorname{Obj}(\mathfrak{S})$ and $f \in \mathfrak{K}$ with $\Phi(s) = \operatorname{dom}(f)$, there exist $g \in \mathfrak{S}$ and $h \in \mathfrak{K}$ such that $\Phi(g) = h \circ f$.

We say that a subcategory $\mathfrak{F}$ of $\mathfrak{K}$ is *dominating* if the inclusion functor $\Phi\colon \mathfrak{F} \to \mathfrak{K}$ is dominating.

Theorem

Let $\mathscr{W} \subseteq \mathrm{Obj}(\mathbf{V})$ *and let* $\Phi \colon \mathfrak{S} \to \mathfrak{K}$ *be a dominating functor. Define* $\mathscr{U}$ *to be the class of all sequences* $\vec{s} \colon \omega \to \mathfrak{S}$ *satisfying* $\lim(\Phi \circ \vec{s}) \in \mathscr{W}$*. Then Odd has a winning strategy in* $\mathrm{BM}(\mathfrak{K}, \mathscr{W})$ *if and only if he has a winning strategy in* $\mathrm{BM}(\mathfrak{S}, \mathscr{U})$*. The same applies to Eve.*

Example I

Let $\mathfrak{K} = \mathscr{T}^+(X)$, where X is a compact Hausdorff space. Let **V** be the family of all G_δ subsets of X. Define

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Theorem (Oxtoby)

Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, \mathscr{W})$ *if and only if* X *contains a dense completely metrizable subspace.*

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A generic object, if exists, is unique up to isomorphism.

Theorem

Let W be a generic object and let X be a $\mathbf{V}$-object of the form $X = \lim \{X_n\}_{n\in\omega}$ for some sequence $\{X_n\}_{n\in\omega}$ in $\mathfrak{K}$. Then

$$\mathbf{V}(X, W) \neq \emptyset.$$

Fraïssé limits

Definition

A Fraïssé class is a class $\mathscr{F}$ of finitely generated models if a fixed first order language, satisfying the following conditions:

1. For every $A, B \in \mathscr{F}$ there is $D \in \mathscr{F}$ such that both A and B can be embedded into D.
2. For every embeddings $f\colon C \to A$, $g\colon C \to B$ with $C, A, B \in \mathscr{F}$, there exist $E \in \mathscr{F}$ and embeddings $f'\colon A \to E$, $g'\colon B \to E$ such that
$$f' \circ f = g' \circ g.$$
3. $\mathscr{F}$ has countably many isomorphic types.

Theorem (Fraïssé)

Let $\mathscr{F}$ be a Fraïssé class. Then there exists a unique countably generated model U of the same language as that of $\mathscr{F}$, having the following properties:

1. *Every $E \in \mathscr{F}$ embeds into U.*
2. *For every finite set $S \subseteq U$ there is an embedding $e \colon E \to U$ such that $E \in \mathscr{F}$ and $S \subseteq e[E]$.*
3. *For every $E \in \mathscr{F}$, for every two embeddings $f, g \colon E \to U$ there exists an automorphism $h \colon U \to U$ such that $h \circ f = g$.*

Theorem

Let $\mathfrak{K}$ be a category whose objects form a Fraïssé class $\mathscr{F}$ and arrows are embeddings. Let U be the Fraïssé limit of $\mathscr{F}$. Then Odd has a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$.

Example II

Theorem

Let $\mathfrak{K}$ be the category of all nonempty compact metrizable spaces with continuous surjections and assume that the Banach-Mazur game is played with reversed arrows. Then the $\mathfrak{K}$-generic compact space is the Cantor set.

Example III

Theorem

Let $\mathfrak{K}$ be the category of all finite metric spaces with isometric embeddings. Then the $\mathfrak{K}$-generic object is the Urysohn universal metric space.

References

- W. Kubiś, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Ann. Pure Appl. Logic 165 (2014) 1755–1811
- W. Kubiś, *Metric-enriched categories and approximate limits*, preprint, `http://arxiv.org/abs/1210.6506`