Non-separable growth of ω with strictly positive measure

Piotr Drygier

joint work with Grzegorz Plebanek

Instytut Matematyczny Uniwersytetu Wrocławskiego

Hejnice, 2015

We are interested in the 0-dimensional compactifications of ω which are denoted by $\gamma\omega$.

We are interested in the 0-dimensional compactifications of ω which are denoted by $\gamma\omega$. Such compactification comes from Stone representation of some Boolean algebra $\mathfrak{A}$ containing finite subsets of ω.

We are interested in the 0-dimensional compactifications of ω which are denoted by $\gamma\omega$. Such compactification comes from Stone representation of some Boolean algebra $\mathfrak{A}$ containing finite subsets of ω.

Definition. A growth of ω is a compact set K such that there exists a compactification $\gamma\omega$ and $K \simeq \gamma\omega \setminus \omega$.

We are interested in the 0-dimensional compactifications of ω which are denoted by $\gamma\omega$. Such compactification comes from Stone representation of some Boolean algebra $\mathfrak{A}$ containing finite subsets of ω.

Definition. A growth of ω is a compact set K such that there exists a compactification $\gamma\omega$ and $K \simeq \gamma\omega \setminus \omega$.

Definition. A measure μ is strictly positive if $\mu(U) > 0$ for any non-empty open set.

Fact. Every separable compactum is a growth of ω.

Fact. Every separable compactum is a growth of ω.

Theorem (Parovičenko). Under CH every compact space of weight at most $\mathfrak{c}$ is a growth of ω.

Fact. Every separable compactum is a growth of ω.

Theorem (Parovičenko). Under CH every compact space of weight at most $\mathfrak{c}$ is a growth of ω.

Corollary. Under CH a Stone space of the measure algebra is a non-separable growth of ω carrying strictly positive measure.

Fact. Every separable compactum is a growth of ω.

Theorem (Parovičenko). Under CH every compact space of weight at most $\mathfrak{c}$ is a growth of ω.

Corollary. Under CH a Stone space of the measure algebra is a non-separable growth of ω carrying strictly positive measure.

Theorem (Dow & Hart). Under OCA the Stone space of the measure algebra is not a growth of ω.

Fact. Every separable compactum is a growth of ω.

Theorem (Parovičenko). Under CH every compact space of weight at most $\mathfrak{c}$ is a growth of ω.

Corollary. Under CH a Stone space of the measure algebra is a non-separable growth of ω carrying strictly positive measure.

Theorem (Dow & Hart). Under OCA the Stone space of the measure algebra is not a growth of ω.

Question. Does there exists in ZFC a non-separable growth of ω carrying a strictly positive measure?

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Summary

The aim is to construct the compactification $\gamma\omega$ of ω such that its remainder (i.e. $\gamma\omega \setminus \omega$) is non-separable and supports the strictly positive measure.

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Summary

The aim is to construct the compactification $\gamma\omega$ of ω such that its remainder (i.e. $\gamma\omega \setminus \omega$) is non-separable and supports the strictly positive measure.

Notation.

- λ denotes a standard product measure on 2^ω,

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Summary

The aim is to construct the compactification $\gamma\omega$ of ω such that its remainder (i.e. $\gamma\omega \setminus \omega$) is non-separable and supports the strictly positive measure.

Notation.

- λ denotes a standard product measure on 2^ω,
- $\mathcal{E} = \{A \in 2^\omega \colon \lambda(\bar{A}) = 0\}$,

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Summary

The aim is to construct the compactification $\gamma\omega$ of ω such that its remainder (i.e. $\gamma\omega \setminus \omega$) is non-separable and supports the strictly positive measure.

Notation.

- λ denotes a standard product measure on 2^ω,
- $\mathcal{E} = \{A \in 2^\omega \colon \lambda(\bar{A}) = 0\}$,
- $\kappa_0 = \mathrm{cov}(\mathcal{E})$

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Summary

The aim is to construct the compactification $\gamma\omega$ of ω such that its remainder (i.e. $\gamma\omega \setminus \omega$) is non-separable and supports the strictly positive measure.

Notation.

- λ denotes a standard product measure on 2^ω,
- $\mathcal{E} = \{A \in 2^\omega \colon \lambda(\bar{A}) = 0\}$,
- $\kappa_0 = \mathrm{cov}(\mathcal{E}) = \min\left\{|\mathcal{U}| \colon \mathcal{U} \subseteq \mathcal{E} \text{ and } \bigcup \mathcal{U} = 2^\omega\right\}$,

Remark. There are constructions (in ZFC) of compactifications of ω with non-separable growth (by Bell, van Mill, Todorčević) but it seems that there is no strictly positive measure on those spaces.

Summary

The aim is to construct the compactification $\gamma\omega$ of ω such that its remainder (i.e. $\gamma\omega \setminus \omega$) is non-separable and supports the strictly positive measure.

Notation.

- λ denotes a standard product measure on 2^ω,
- $\mathcal{E} = \{A \in 2^\omega \colon \lambda(\bar{A}) = 0\}$,
- $\kappa_0 = \mathrm{cov}(\mathcal{E}) = \min\left\{|\mathcal{U}| \colon \mathcal{U} \subseteq \mathcal{E} \text{ and } \bigcup \mathcal{U} = 2^\omega\right\}$,
- $\mathfrak{b}$ denotes the minimal size of the unbounded family $\mathcal{F} \subseteq \omega^\omega$ ordered by $\leq^*$.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Remark. Observe that OCA implies $\mathfrak{b} = \omega_2$. Moreover, we can assume that $\mathfrak{c} = \omega_2$,

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Remark. Observe that OCA implies $\mathfrak{b} = \omega_2$. Moreover, we can assume that $\mathfrak{c} = \omega_2$, so this construction works in the models of set theory where Stone space of measure algebra is not a growth of ω.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Remark. Observe that OCA implies $\mathfrak{b} = \omega_2$. Moreover, we can assume that $\mathfrak{c} = \omega_2$, so this construction works in the models of set theory where Stone space of measure algebra is not a growth of ω.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a Boolean algebra $\mathfrak{A}$ such that *fin* $\subseteq \mathfrak{A} \subseteq P(\omega)$ and a finitely additive probability measure μ on $\mathfrak{A}$ such that

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Remark. Observe that OCA implies $\mathfrak{b} = \omega_2$. Moreover, we can assume that $\mathfrak{c} = \omega_2$, so this construction works in the models of set theory where Stone space of measure algebra is not a growth of ω.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a Boolean algebra $\mathfrak{A}$ such that $\mathit{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ and a finitely additive probability measure μ on $\mathfrak{A}$ such that

- $\mathfrak{A}/\mathit{fin}$ is not σ-centered,

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Remark. Observe that OCA implies $\mathfrak{b} = \omega_2$. Moreover, we can assume that $\mathfrak{c} = \omega_2$, so this construction works in the models of set theory where Stone space of measure algebra is not a growth of ω.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a Boolean algebra $\mathfrak{A}$ such that $fin \subseteq \mathfrak{A} \subseteq P(\omega)$ and a finitely additive probability measure μ on $\mathfrak{A}$ such that

- $\mathfrak{A}/fin$ is not σ-centered,
- $\mu(A) = 0$ if and only if A is finite

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a compactification $\gamma\omega$ of the set of natural numbers such that its remainder $\gamma\omega \setminus \omega$ is non-separable and carries a strictly positive regular probability Borel measure.

Remark. Observe that OCA implies $\mathfrak{b} = \omega_2$. Moreover, we can assume that $\mathfrak{c} = \omega_2$, so this construction works in the models of set theory where Stone space of measure algebra is not a growth of ω.

Theorem (D., Plebanek). Assume $\mathrm{cof}[\kappa_0]^{\leq\omega} \leq \mathfrak{b}$. There exists a Boolean algebra $\mathfrak{A}$ such that $fin \subseteq \mathfrak{A} \subseteq P(\omega)$ and a finitely additive probability measure μ on $\mathfrak{A}$ such that

- ${}^{\mathfrak{A}}/_{fin}$ is not σ-centered,
- $\mu(A) = 0$ if and only if A is finite (μ is almost strictly positive)

Sketch of the proof

Sketch of the proof

Denote $\kappa = \mathrm{cof}[\kappa_0]^{\leq\omega}$.

Sketch of the proof

Denote $\kappa = \mathsf{cof}[\kappa_0]^{\leq\omega}$. We shall prove the theorem by inductive construction, gradually enlarging Boolean algebras and extending measures.

Sketch of the proof

Denote $\kappa = \mathrm{cof}[\kappa_0]^{\leq\omega}$. We shall prove the theorem by inductive construction, gradually enlarging Boolean algebras and extending measures. For any $X \subseteq \omega$ we have

$$\mathfrak{A}[X] = \big\{(A \cap X) \cup (A' \setminus X) \colon A, A' \in \mathfrak{A}\big\}.$$

Sketch of the proof

Denote $\kappa = \mathrm{cof}[\kappa_0]^{\leq\omega}$. We shall prove the theorem by inductive construction, gradually enlarging Boolean algebras and extending measures. For any $X \subseteq \omega$ we have

$$\mathfrak{A}[X] = \big\{(A \cap X) \cup (A' \setminus X) \colon A, A' \in \mathfrak{A}\big\}.$$

$$\mu_*(Z) = \sup\{\mu(A) \colon A \in \mathfrak{A}, A \subseteq Z\}$$

$$\mu^*(Z) = \inf\{\mu(A) \colon A \in \mathfrak{A}, A \supseteq Z\} \qquad \text{for } Z \subseteq \omega.$$

Sketch of the proof

Denote $\kappa = \mathrm{cof}[\kappa_0]^{\leq\omega}$. We shall prove the theorem by inductive construction, gradually enlarging Boolean algebras and extending measures. For any $X \subseteq \omega$ we have

$$\mathfrak{A}[X] = \big\{(A \cap X) \cup (A' \setminus X) \colon A, A' \in \mathfrak{A}\big\}.$$

$$\mu_*(Z) = \sup\{\mu(A) \colon A \in \mathfrak{A}, A \subseteq Z\}$$

$$\mu^*(Z) = \inf\{\mu(A) \colon A \in \mathfrak{A}, A \supseteq Z\} \qquad \text{for } Z \subseteq \omega.$$

We can exetend measure μ on $\mathfrak{A}$ onto a mesure $\tilde{\mu}$ on $\mathfrak{A}[X]$ by setting

$$\tilde{\mu}\big((A \cap X) \cup (A' \setminus X)\big) =$$

Sketch of the proof

Denote $\kappa = \mathrm{cof}[\kappa_0]^{\leq\omega}$. We shall prove the theorem by inductive construction, gradually enlarging Boolean algebras and extending measures. For any $X \subseteq \omega$ we have

$$\mathfrak{A}[X] = \big\{(A \cap X) \cup (A' \setminus X) \colon A, A' \in \mathfrak{A}\big\}.$$

$$\mu_*(Z) = \sup\{\mu(A) \colon A \in \mathfrak{A}, A \subseteq Z\}$$

$$\mu^*(Z) = \inf\{\mu(A) \colon A \in \mathfrak{A}, A \supseteq Z\} \qquad \text{for } Z \subseteq \omega.$$

We can exetend measure μ on $\mathfrak{A}$ onto a mesure $\tilde{\mu}$ on $\mathfrak{A}[X]$ by setting

$$\tilde{\mu}\big((A \cap X) \cup (A' \setminus X)\big) = \mu_*(A \cap X) + \mu^*(A' \setminus X).$$

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,
- μ_ξ is almost strictly positive,

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,
- μ_ξ is almost strictly positive,
- μ_ν extends μ_ξ for $\xi < \nu$.

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,
- μ_ξ is almost strictly positive,
- μ_ν extends μ_ξ for $\xi < \nu$.

We begin with a $\mathfrak{A}_0$ which is countable Boolean algebra with a measure almost strictly positive.

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,
- μ_ξ is almost strictly positive,
- μ_ν extends μ_ξ for $\xi < \nu$.

We begin with a $\mathfrak{A}_0$ which is countable Boolean algebra with a measure almost strictly positive.

On each step $\alpha + 1$ of the induction we add to the algebra new set $X_{\alpha+1}$ such that

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,
- μ_ξ is almost strictly positive,
- μ_ν extends μ_ξ for $\xi < \nu$.

We begin with a $\mathfrak{A}_0$ which is countable Boolean algebra with a measure almost strictly positive.

On each step $\alpha + 1$ of the induction we add to the algebra new set $X_{\alpha+1}$ such that

- $X_{\alpha+1}$ kills a chosen countable set to be dense in Stone$(\mathfrak{A}_0)$,

We construct an increasing sequence of algebras $(\mathfrak{A}_\xi, \mu_\xi)$ such that

- $|\mathfrak{A}_\xi| < \mathfrak{b}$,
- μ_ξ is almost strictly positive,
- μ_ν extends μ_ξ for $\xi < \nu$.

We begin with a $\mathfrak{A}_0$ which is countable Boolean algebra with a measure almost strictly positive.

On each step $\alpha + 1$ of the induction we add to the algebra new set $X_{\alpha+1}$ such that

- $X_{\alpha+1}$ kills a chosen countable set to be dense in Stone($\mathfrak{A}_0$),
- we should be able to extend the a.s.p. measure μ_α to a.s.p. $\tilde{\mu}$ on $\mathfrak{A}[X_{\alpha+1}]$.

Finally we take $\mathfrak{A} = \bigcup_{\xi<\kappa} \mathfrak{A}_\xi$. The measure μ on $\mathfrak{A}$ is the extension of all μ_ξ.

Finally we take $\mathfrak{A} = \bigcup_{\xi<\kappa} \mathfrak{A}_\xi$. The measure μ on $\mathfrak{A}$ is the extension of all μ_ξ.

Why algebra $\mathfrak{A}$ satisfies the condition that the Stone space of it has the desired properties?

Finally we take $\mathfrak{A} = \bigcup_{\xi<\kappa} \mathfrak{A}_\xi$. The measure μ on $\mathfrak{A}$ is the extension of all μ_ξ.

Why algebra $\mathfrak{A}$ satisfies the condition that the Stone space of it has the desired properties?

Take any countable set $\{p_i \colon i \in \omega\} \subseteq \mathrm{Stone}(\mathfrak{A})$.

Finally we take $\mathfrak{A} = \bigcup_{\xi<\kappa} \mathfrak{A}_\xi$. The measure μ on $\mathfrak{A}$ is the extension of all μ_ξ.

Why algebra $\mathfrak{A}$ satisfies the condition that the Stone space of it has the desired properties?

Take any countable set $\{p_i \colon i \in \omega\} \subseteq \mathrm{Stone}(\mathfrak{A})$. On step α of the induction we have added a set X which generates an open non-empty set which does not contain the points p_i.

Finally we take $\mathfrak{A} = \bigcup_{\xi<\kappa} \mathfrak{A}_\xi$. The measure μ on $\mathfrak{A}$ is the extension of all μ_ξ.

Why algebra $\mathfrak{A}$ satisfies the condition that the Stone space of it has the desired properties?

Take any countable set $\{p_i \colon i \in \omega\} \subseteq \mathrm{Stone}(\mathfrak{A})$. On step α of the induction we have added a set X which generates an open non-empty set which does not contain the points p_i.

Question

Does there exist a ZFC construction of a non-separable growth of ω carrying a strictly positive measure?