Inscribing Baire-nonmeasurable sets

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T - uncountable Polish space,
$\mathbb{I}$ - a σ-ideal of subsets of T with Borel base.

Definition

Let $N \subseteq X \subseteq T$. We say that the set N is *completely* $\mathbb{I}$*-nonmeasurable* in X if

$$(\forall A \in \mathrm{Borel})(A \cap X \notin \mathbb{I} \to (A \cap N \notin \mathbb{I}) \wedge (A \cap (X \setminus N) \notin \mathbb{I})).$$

Remark

- $N \subseteq \mathbb{R}$ is completely $\mathbb{L}$-nonmeasurable if $\lambda_*(N) = 0$ and $\lambda_*(\mathbb{R} \setminus N) = 0$.
- The definition of completely $\mathbb{K}$-nonmeasurability is equivalent to the definition of completely Baire-nonmeasurability.
- N is completely $[\mathbb{R}]^\omega$-nonmeasurable iff N is a Bernstein set.

Definition

The ideal $\mathbb{I} \subseteq P(T)$ have *the hole property* if for every set $A \subseteq T$ there is a $\mathbb{I}$-minimal Borel set B containing A i.e. $B \setminus A \in \mathbb{I}$ and if $A \subseteq C$ and C is Borel then $B \setminus C \in \mathbb{I}$.
In such case we will write

$$[A]_{\mathbb{I}} = B \quad \text{and} \quad]A[_{\mathbb{I}} = T \setminus [T \setminus A]_{\mathbb{I}}.$$

Remark

Every c.c.c. σ-ideal with Borel base have the hole property.

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N is completely $\mathbb{I}$-nonmeasurable in X iff

$$[N]_{\mathbb{I}} = [X]_{\mathbb{I}} \text{ and }]N[_{\mathbb{I}} = \emptyset.$$

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T uncountable Polish space,
$\mathbb{K}$ - the ideal of meager sets.

Theorem (Cichoń, Morayne, Rałowski, Ryll-Nardzewski, Ż, 2007)

Let $\mathcal{A} \subseteq \mathbb{K}$ be a partition of a subset of T. Then we can find a subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B}$ is completely $\mathbb{K}$-nonmeasurable in $\bigcup \mathcal{A}$.

Theorem (Gitik, Shelah 2001)

Let $(A_n : n \in \omega)$ *be a sequence of subsets of* $\mathbb{R}$.
Then we can find a sequence $(B_n : n \in \omega)$ *such that*

1. $B_n \cap B_m = \emptyset$ *for* $n \neq m$,
2. $B_n \subseteq A_n$,
3. $\lambda^*(A_n) = \lambda^*(B_n)$, *where* λ^* *is outer Lebesgue measure.*

Theorem

Assume that $\mathcal{A} \subseteq \mathbb{K}$ *is a partition of* T. *Let* $\mathcal{A}_n \subseteq \mathcal{A}$ *for* $n \in \omega$. *Then there exists* $\mathcal{B}_n \subseteq \mathcal{A}_n$ *such that*

1. $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ *for* $n \neq m$,
2. $\mathcal{B}_n \subseteq \mathcal{A}_n$,
3. $[\bigcup \mathcal{B}_n]_{\mathbb{K}} = [\bigcup \mathcal{A}_n]_{\mathbb{K}}$.

Theorem

Assume that $\mathcal{A} \subseteq \mathbb{K}$ *is a partition of a subset of* T *and* $\bigcup \mathcal{A} \notin \mathbb{K}$. *Let* $\mathcal{A}_n \subseteq \mathcal{A}$ *for* $n \in \omega$. *Then there exists* $\mathcal{B} \subseteq \mathcal{A}$ *such that*

1. $\bigcup \mathcal{B} \notin \mathbb{K}$,
2. $]\bigcup \mathcal{B} \cap \bigcup \mathcal{A}_n[_{\mathbb{K}} = \emptyset$ *for every* $n \in \omega$.

Theorem (Gitik, Shelah, 1989)

If I is a σ-ideal on κ, then $P(\kappa)/I$ is not isomorphic to the Cohen algebra.

Theorem

Assume that $\mathcal{A} \subseteq \mathbb{K}$ is a partition of a subset of T. Let $\mathcal{A}_n \subseteq \mathcal{A}$ for $n \in \omega$. Then there exists $\mathcal{B} \subseteq \mathcal{A}$ such that

1. $[\bigcup \mathcal{B}]_{\mathbb{K}} = [\bigcup \mathcal{A}]_{\mathbb{K}}$,
2. $]\bigcup \mathcal{B} \cap \bigcup \mathcal{A}_n[_{\mathbb{K}} = \emptyset$ *for every* $n \in \omega$.

Theorem (Alaoglu, Erdös, 1950)

For every cardinal κ

$$((\kappa\colon \omega, \omega_1) \to \omega_1) \leftrightarrow ((\kappa\colon 1, \omega_1) \to \omega_1).$$

Lemma

Assume that $\{I_n\}_{n\in\omega}$ *is a family of* σ*-additive ideals on* κ *which are not c.c.c. Then there exists a family* $\{X_\alpha\}_{\alpha<\omega_1} \subseteq P(\kappa)$ *such that*

1. $(\forall\alpha < \omega_1)(\forall n \in \omega)(X_\alpha \notin I_n)$
2. $(\forall\alpha, \beta < \omega_1)(\alpha \neq \beta \rightarrow X_\alpha \cap X_\beta = \emptyset)$.

Theorem

Assume that $\mathcal{A} \subseteq \mathbb{K}$ *is a partition of* T. *Let* $\mathcal{A}_n \subseteq \mathcal{A}$ *for* $n \in \omega$. *Then there exists* $\mathcal{B}_n \subseteq \mathcal{A}_n$ *such that*

1. $\mathcal{B}_n \cap \mathcal{B}_m = \emptyset$ *for* $n \neq m$,
2. $\mathcal{B}_n \subseteq \mathcal{A}_n$,
3. $[\bigcup \mathcal{B}_n]_{\mathbb{K}} = [\bigcup \mathcal{A}_n]_{\mathbb{K}}$.

Thank You for Your Attention

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