Ultrafilters and Michael Spaces

Arturo Martínez-Celis

Centro de Ciencias Matemáticas, UNAM
Hejnice, 2015
Joint work with Michael Hrušák

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Examples.

- Compact spaces are productively Lindelöf.
- σ-compact spaces are productively Lindelöf.

What kind of metric Lindelöf spaces can be productively Lindelöf?

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E. Michael

CH implies that metric productively Lindelöf spaces are σ-compact.

Under CH, there is a Lindelöf space which has a non-Lindelöf product with ω^ω.

Definition

A Lindelöf space X is a *Michael space* if its product with the space of the irrational numbers is not a Lindelöf space.

Michael space problem

Is there a Michael space?

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Under $\mathfrak{b} = \omega_1$ (E. Michael) or if $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$ (J. Moore) they do!. For the general case the answer is still unknown.

Definition

Let $\mathfrak{U}$ be a filter over ω and $f, g \in \omega^\omega$, we say that $f \leq_{\mathfrak{U}} g$ if $\{n \in \omega : f(n) \leq g(n)\} \in \mathfrak{U}$.

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If $\mathfrak{U}$ is an ultrafilter then $\leq_{\mathfrak{U}}$ is a total order and if

$$\mathfrak{d}_{\mathfrak{U}} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is } \leq_{\mathfrak{U}} \text{-cofinal}\}$$

then $\mathfrak{d}_{\mathfrak{U}}$ is a regular cardinal and $\mathfrak{b} \leq \mathfrak{d}_{\mathfrak{U}} \leq \mathfrak{d}$.

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For every $A \subseteq \omega^\omega$ we define

$$\mathfrak{d}_\mathfrak{u}(A) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq A \text{ is } \leq_\mathfrak{U} \text{-cofinal in } A)\}.$$

Definition

A filter $\mathcal{U}$ over ω is a Michael filter if for every compact set $K \subseteq \omega^\omega$, if $\mathfrak{d}_{\mathcal{U}}(K) > \omega$ then $\mathfrak{d}_{\mathcal{U}}(K) \geq \mathfrak{d}_{\mathcal{U}}$

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Examples.... well, example

The Frechet Filter.

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The Frechet Filter.

Theorem

If there is a Michael ultrafilter, then there is a Michael space.

Let $\{f_\alpha\}_{\alpha\in\mathfrak{d}_\mathcal{U}}$ be a strictly $\le_\mathcal{U}$-increasing $\le_\mathcal{U}$-unbounded sequence and for each $\alpha\in\mathfrak{d}_\mathcal{U}$ define

$$X_\alpha = \{f\in\omega^\omega : \exists\beta<\alpha(f\le_\mathcal{U} f_\beta)\}$$

Properties of X_α

- if $\alpha<\beta$ then $X_\alpha \subsetneq X_\beta$,
- for every compact $K\subseteq\omega^\omega$, the least ordinal γ such that $K\subseteq X_\gamma$ has finite or countable cofinality,

The last one follows from the fact that if γ is not a succesor ordinal, then one can construct an internal $\le_\mathcal{U}$-unbounded family of cardinality cof(γ). The rest follows from the following.

Theorem (J. Moore)

If there exist a sequence $\langle X_\alpha \rangle_{\alpha \leq \kappa}$ of subsets of irrational numbers such that

- *if $\alpha < \beta$ then $X_\alpha \subsetneq X_\beta$,*
- *for every compact $K \subseteq \omega^\omega$, the least ordinal γ such that $K \subseteq X_\gamma$ has finite or countable cofinality,*

then there exists a Michael space.

Therefore, the existence of a Michael Ultrafilter implies the existence of a Michael space.

The easiest way to construct a Michael ultrafilter is to construct an ultrafilter with $\mathfrak{d}_{\mathfrak{U}} = \omega_1$. As $\mathfrak{d}_{\mathfrak{U}} \leq \mathfrak{d}$ we have the following fact

Easy fact

[$\mathfrak{d} = \omega_1$] Every Ultrafilter is a Michael ultrafilter.

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They are not!

It is consistent that there is an ultrafilter which is not a Michael ultrafilter.

In Miller's model, every *p*-point has character ω_1 and $\mathfrak{d} = \mathfrak{d}_{\mathfrak{U}} = \omega_2$. The only thing left to do is to show that there is a compact set K such that $\mathfrak{d}_{\mathfrak{U}}(K) = \omega_1$. For every $A \subseteq \omega$ consider its increasing ennumeration $\{a_n : n \in \omega\}$ and define $\varphi_A \in \omega^\omega$ as

$$\varphi_A(k) = \begin{cases} a_0 & \text{if } k = a_0 \\ a_{k+1} - a_k & \text{if } k = a_{n+1} \\ 0 & \text{in other case} \end{cases}$$

and let $K = \{\varphi_A : A \subseteq \omega\}$. K is a compact space and if $A \subseteq B$ then $\varphi_A|A \geq \varphi_B|A$.
Thefore the following hold:

- If A is a pseudointersection of $\{A_i\}_{i\in\omega}$ then $\varphi_A|A \geq^* \varphi_{A_i}|A$,
- If $\mathcal{B}$ is a base for the ultrafilter $\mathfrak{U}$, then $\{\varphi_B : B \in \mathcal{B}\}$ is a $\leq_{\mathfrak{U}}$-cofinal sequence.

As a consequence, in Miller's model $\mathfrak{d}_{\mathfrak{U}}(K) = \omega_1$.

Corollary

In Miller's model, every p-point fails to be a Michael ultrafilter.

I still don't know if there is a Michael ultrafilter in Miller's Model.

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Theorem (Canjar)

Selective ultrafilters exist generically if and only if $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

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Theorem (Canjar)

Selective ultrafilters exist generically if and only if $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$.

Theorem

If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ *and* $\mathfrak{c}$ *is a regular cardinal, then Michael ultrafilters exists generically.*

Theorem

If $\mathfrak{t} = \mathfrak{h}$, *then* $P_\omega/\mathrm{FIN} \Vdash$ *"*$\mathcal{U}_{gen}$ *is a Michael ultrafilter"*

Theorem

If $\mathfrak{t} = \mathfrak{h}$, then $P_\omega/\mathrm{FIN} \Vdash$ "$\mathcal{U}_{gen}$ is a Michael ultrafilter"

Proof. P_ω/FIN doesn't add reals and collapses $\mathfrak{c}$ to $\mathfrak{h}$. So, if the theorem is false, there should be a ground model compact K, an uncountable subcollection $\{f_\alpha\}_{\alpha\in\lambda}$ with $\lambda < \mathfrak{h}$, and X such that

$$X \Vdash \text{"}\{f_\alpha\}_{\alpha\in\lambda} \text{ is } \leq_{\mathfrak{U}_{gen}} \text{ cofinal in } K\text{"}$$

Recall that the Frechet filter is Michael, therefore there is f not dominated by $\{f_\alpha\}_{\alpha\in\lambda}$. If $A_\alpha = \{n : f(n) \geq f_\alpha(n)\}$, then $\{A_\alpha\}$ is a centered family. This family has a pseudointersection contained in X!

The big question

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A not so big... but still big question

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Is there a Michael Space on the Mathias/Laver model? Is there a Michael Ultrafilter on the Mathias/Laver model?

	Michael Space	Michael Ultrafilter
ZFC	?	?
Cohen	Yes	Yes
Random	Yes	Yes
Hechler	Yes	Yes
Mathias	?	?
Laver	?	?
Miller	Yes	?
Sacks	Yes	Yes

More questions

$P_\omega/\mathrm{FIN} \Vdash$ "$\mathcal{U}_{gen}$ is a Michael ultrafilter" ?

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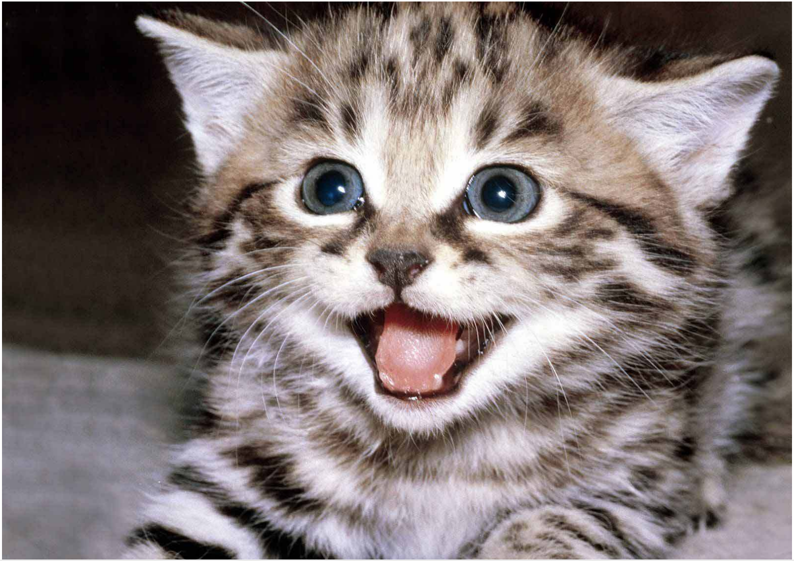
$P_\omega/\mathrm{FIN} \Vdash$ "$\mathcal{U}_{gen}$ is a Michael ultrafilter" ?

Last, but not less important

Are there another example of a Michael filter? Can we classify Borel Michael filters?

Thank you for your attention! Here’s a picture of a cat



Not a cat person?