Sacks dense ideals and Marczewski type null ideals

Wolfgang Wohofsky

Vienna University of Technology (TU Wien)

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The real numbers: topology, measure, algebraic structure

The real numbers ("the reals")

- $\mathbb{R}$, the classical real line
- 2^ω, the Cantor space (totally disconnected, compact)

Structure on the reals:

- natural topology (intervals/basic clopen sets form a basis)
- standard (Lebesgue) measure
- group structure
  - $(2^\omega, +)$ is a topological group, with $+$ bitwise modulo 2
- Two translation-invariant σ-ideals
  - meager sets $\mathcal{M}$
  - measure zero sets $\mathcal{N}$

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Strong measure zero sets

For an interval $I \subseteq \mathbb{R}$, let $\lambda(I)$ denote its length.

Definition (well-known)

A set $X \subseteq \mathbb{R}$ is (Lebesgue) measure zero ($X \in \mathcal{N}$) if
for each positive real number $\varepsilon > 0$
there is a sequence of intervals $(I_n)_{n<\omega}$ of total length $\sum_{n<\omega} \lambda(I_n) \leq \varepsilon$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Definition (Borel; 1919)

A set $X \subseteq \mathbb{R}$ is strong measure zero ($X \in \mathcal{SN}$) if
for each sequence of positive real numbers $(\varepsilon_n)_{n<\omega}$
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Equivalent characterization of strong measure zero sets

For $Y, Z \subseteq 2^\omega$, let $Y + Z = \{y + z : y \in Y, z \in Z\}$.

Key Theorem (Galvin,Mycielski,Solovay; 1973)

A set $Y \subseteq 2^\omega$ is strong measure zero if and only if for every meager set $M \in \mathcal{M}$, $Y + M \neq 2^\omega$.

Note that $Y + M \neq 2^\omega$ if and only if Y can be "translated away" from M, i.e., there exists a $t \in 2^\omega$ such that $(Y + t) \cap M = \emptyset$.

Key Definition

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

$\mathcal{J}^\star$ is the collection of "$\mathcal{J}$-shiftable sets", i.e., $Y \in \mathcal{J}^\star$ if Y can be translated away from every set in $\mathcal{J}$.

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$$\mathcal{SN} = \mathcal{M}^\star$$

Replacing $\mathcal{M}$ by $\mathcal{N}$ yields a notion *dual to strong measure zero*:

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Borel Conjecture + dual Borel Conjecture

Definition

The Borel Conjecture (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e., $\mathcal{SN} = \mathcal{M}^\star = [2^\omega]^{\leq \aleph_0}$.

- Con(BC), actually BC holds in the Laver model (Laver, 1976)

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Theorem (Goldstern, Kellner, Shelah, W.; 2011)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).

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Marczewski Borel Conjecture (MBC)

Assume that $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ is a translation-invariant σ-ideal. Recall that

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

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The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\aleph_0}$.

The Marczewski ideal s_0 is the collection of all $Z \subseteq 2^\omega$ such that for each perfect set P, there exists a perfect subset $Q \subseteq P$ with $Q \cap Z = \emptyset$.

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Sacks dense ideals

Unlike BC and dBC, the status of MBC under CH is unclear...

- Is MBC (i.e., ${s_0}^\star = [2^\omega]^{\leq\aleph_0}$) consistent with CH?
- Or does CH even imply MBC?

To investigate the situation under CH, I introduced the following notion:

Definition

A collection $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ is a Sacks dense ideal if

- $\mathcal{I}$ is a σ-ideal,
- $\mathcal{I}$ is *translation-invariant*,
- $\mathcal{I}$ is dense in Sacks forcing, more explicitly, for each perfect $P \subseteq 2^\omega$, there is a perfect subset Q in the ideal, i.e., $\exists Q \subseteq P$, $Q \in \mathcal{I}$.

Lemma ("Main Lemma")

Assume CH. Let $\mathcal{I}$ be a Sacks dense ideal. Then ${s_0}^\star \subseteq \mathcal{I}$.

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In other words: ${s_0}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I}$ is a Sacks dense ideal$\}$.

Can we (consistently) find many Sacks dense ideals under CH?

$\mathcal{M}$ $\quad$ $\mathcal{N}$ $\quad$ are Sacks dense ideals

$\mathcal{M} \supsetneq \mathcal{M} \cap \mathcal{N} \subsetneq \mathcal{N}$

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$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT...

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$\mathcal{SN}$ is NOT a Sacks dense ideal, BUT...

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$$\bigcap\{\mathcal{I}_f : f \in \omega^\omega\} \quad \subseteq \text{null-additive} \subseteq \mathcal{SN} \cap \mathcal{SM}$$

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$\exists$ uncount. $Y \in \bigcap\{\mathcal{I}_\alpha : \alpha \in \omega_1\}$, for any $\aleph_1$-sized family of $\mathcal{I}_\alpha$'s

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$$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p\, \exists q \le p \qquad\qquad |[q] \cap Y| \le \aleph_0$$

Definition

$Y \in s_0^{\text{trans}} :\Longleftrightarrow \forall p\, \exists q \le p\ \forall t \in 2^\omega\ \ |(t + [q]) \cap Y| \le \aleph_0$

Theorem (using s_0^{trans})

- Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha \in \omega_1} \mathcal{I}_\alpha$.
- Under CH, we can construct the set Y in such a way that $Y \in s_0^{\text{trans}}$.
- $Y \in s_0^{\text{trans}}$ implies that there is a Sacks dense ideal $\mathcal{J}$ with $Y \notin \mathcal{J}$.

Question

Does $[2^\omega]^{\le \aleph_0} = \bigcap \{\mathcal{I} : \mathcal{I} \text{ is S.d.i.}\}$ (at least consistently) hold under CH?

If yes, MBC (i.e., $s_0^\star = [2^\omega]^{\le \aleph_0}$) follows from CH (Con(MBC+CH), resp.).

$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p \, \exists q \le p \qquad\qquad |[q] \cap Y| \le \aleph_0$

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If yes, MBC (i.e., $s_0{}^\star = [2^\omega]^{\le \aleph_0}$) follows from CH (Con(MBC+CH), resp.).

$Y \in s_0 \quad :\Longleftrightarrow \quad \forall p\, \exists q \le p \qquad\qquad |[q] \cap Y| \le \aleph_0$

Definition

$Y \in s_0^{\mathrm{trans}} :\Longleftrightarrow \forall p\, \exists q \le p\ \forall t \in 2^\omega\ |(t + [q]) \cap Y| \le \aleph_0$

Theorem (using s_0^{trans})

- Let $\{\mathcal{I}_\alpha : \alpha < \omega_1\}$ be an $\aleph_1$-sized family of Sacks dense ideals. Then there exists an uncountable set $Y \in \bigcap_{\alpha \in \omega_1} \mathcal{I}_\alpha$.
- Under CH, we can construct the set Y in such a way that $Y \in s_0^{\mathrm{trans}}$.
- $Y \in s_0^{\mathrm{trans}}$ implies that there is a Sacks dense ideal $\mathcal{J}$ with $Y \notin \mathcal{J}$.

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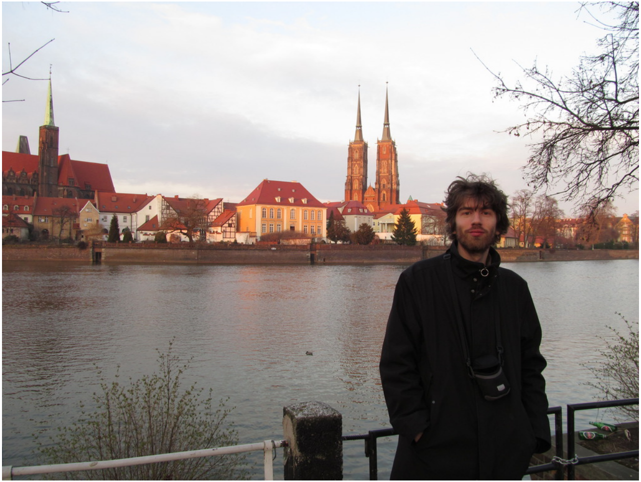
Thank you for your attention and enjoy the Winter School. . .



Myself in Wrocław