Sequences of continuous and semicontinuous functions

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All spaces are assumed to be Hausdorff and infinite.

Diagrams hold for perfectly normal space.

Ohta–Sakai's properties

Ohta H. and Sakai M., *Sequences of semicontinuous functions accompanying continuous functions*, Topology Appl. **156** (2009), 2683-2906.

USC	LSC
USC_s	LSC_s
USC_m	LSC_m

USC	X has property USC, if whenever $\langle f_n : n \in \omega \rangle$ of **upper** semicontinuous functions with values in $[0,1]$ converges to zero, there is $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero such that $f_n \le g_n$ for any $n \in \omega$.
USC_s	X has property USC_s, if whenever $\langle f_n : n \in \omega \rangle$ of upper semicontinuous functions with values in $[0, 1]$ converges to zero, there is $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero and an increasing sequence $\{n_m\}_{m=0}^{\infty}$ such that $f_{n_m} \le g_m$ for any $m \in \omega$.
USC_m	X has property USC_m, if whenever $\langle f_n : n \in \omega \rangle$ of upper semicontinuous functions with values in $[0, 1]$ converges to zero and $f_{n+1} \le f_n$, $n \in \omega$, there is $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero such that $f_n \le g_n$ for any $n \in \omega$.
LSC	X has property LSC, if whenever $\langle f_n : n \in \omega \rangle$ of **lower** semicontinuous functions with values in $[0, 1]$ converges to zero, there is $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero such that $f_n \le g_n$ for any $n \in \omega$.
LSC_s	X has property LSC_s, if whenever $\langle f_n : n \in \omega \rangle$ of lower semicontinuous functions with values in $[0, 1]$ converges to zero, there is $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero and an increasing sequence $\{n_m\}_{m=0}^{\infty}$ such that $f_{n_m} \le g_m$ for any $m \in \omega$.
LSC_m	X has property LSC_m, if whenever $\langle f_n : n \in \omega \rangle$ of lower semicontinuous functions with values in $[0, 1]$ converges to zero and $f_{n+1} \le f_n$, $n \in \omega$, there is $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero such that $f_n \le g_n$ for any $n \in \omega$.

A function f is said to be lower semicontinuous, upper semicontinuous, if for every real number r the set

$f^{-1}((r, \infty)) = \{x \in X : f(x) > r\}$, $f^{-1}((-\infty, r)) = \{x \in X : f(x) < r\}$ is open in a space X, respectively.

$$\boxed{\mathrm{USC} \to \mathrm{USC}_s \to \mathrm{USC}_m}$$

$$\boxed{\mathrm{LSC} \to \mathrm{LSC}_s \to \mathrm{LSC}_m}$$

$$\mathrm{USC}_m \nrightarrow \mathrm{USC}_s$$

$$\textbf{ZFC} + \mathfrak{p} = \mathfrak{b} \vdash \mathrm{USC}_s \nrightarrow \mathrm{USC}$$

Any discrete space satisfies all Ohta–Sakai's properties.

Theorem (H. Ohta – M. Sakai [2009])

(1) *Every compact scattered space has* USC*.*

(2) *Every ordinal with the order topology has* USC*.*

(3) *Every normal countably paracompact* P*-space has* USC*.*

(4) *Every* γ*-set has* USC_s*.*

Theorem (H. Ohta – M. Sakai [2009])

Every separable metrizable space with USC_s *is perfectly meager.*

Theorem (H. Ohta – M. Sakai [2009])

Every normal countably paracompact space has USC_m*, and every space with* USC_m *is countably paracompact.*

Theorem (H. Ohta – M. Sakai [2009])

A topological space X *has* USC_m *if and only if* X *is a* cb*-space.*

Let X be a topological space. A set $A \subseteq X$ is called perfectly meager if for any perfect set $P \subseteq X$ the intersection $A \cap P$ is meager in the subspace P.

A topological space X is called a cb-space if for each real-valued locally bounded function f on X there is a continuous function g such that $|f| \leq g$. (J.G. Horne [1959])

Proposition

For a perfectly normal space X the following are equivalent.

(1) X *possesses* USC.

(2) *For any sequence $\langle f_n : n \in \omega \rangle$ of upper semicontinuous functions on X with values in $[0,1]$ converging to zero, there is a sequence $\langle g_n : n \in \omega \rangle$ of lower semicontinuous functions converging to zero such that $f_n \leq g_n$ for any $n \in \omega$.*

(3) *For any sequence $\langle f_n : n \in \omega \rangle$ of simple upper semicontinuous functions on X with values in $[0,1]$ converging to zero, there is a sequence $\langle g_n : n \in \omega \rangle$ of continuous functions converging to zero such that $f_n \leq g_n$ for any $n \in \omega$.*

(5) *For any sequence $\langle f_n : n \in \omega \rangle$ of upper semicontinuous functions on X with values in $\mathbb{R}$ converging to a function f on X, there is a sequence $\langle g_n : n \in \omega \rangle$ of continuous functions converging to f such that $f_n \leq g_n$ for any $n \in \omega$.*

Similarly for USC_s.

Šupina J., *Notes on modifications of a* wQN*-space,* Tatra Mt. Math. Publ. **58** (2014), 129–136.

Šupina J., *On Ohta–Sakai's properties of a topological space,* to appear.

a set X, $\mathcal{F}, \mathcal{G} \subseteq {}^X\mathbb{R}$, $0 \in \mathcal{F}, \mathcal{G}$

We say that X has **a property** $\mathrm{wED}(\mathcal{F}, \mathcal{G})$, if

(1) for any sequence $\langle f_m : m \in \omega \rangle$ of functions from $\mathcal{F}$ converging to 0,

(2) there are sequences $\langle g_m : m \in \omega \rangle$ and $\langle h_m : m \in \omega \rangle$ of functions from $\mathcal{G}$ converging to 0 and

(3) there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$

such that for any $x \in X$ we have

$$h_m(x) \le f_{n_m}(x) \le g_m(x) \text{ for all but finitely many } m \in \omega.$$

$\mathrm{wED}(\mathcal{F}, \mathcal{G})$ is trivial for $\mathcal{F} \subseteq \mathcal{G}$.

If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $\mathcal{G}_1 \subseteq \mathcal{G}_2$ then $\mathrm{wED}(\mathcal{F}_2, \mathcal{G}_1) \to \mathrm{wED}(\mathcal{F}_1, \mathcal{G}_2)$.

${}^{X}\mathbb{R}$	the family of all real-valued functions on X
${}^{X}[0,1]$	the family of all functions on X with values in $[0,1]$
$\mathrm{C}(X)$	the family of all continuous functions on X
$\mathcal{B}$	the family of all Borel functions on X
$\mathcal{U}$	the family of all upper semicontinuous functions on X
$\mathcal{L}$	the family of all lower semicontinuous functions on X
Const	the family of all constant functions on X

$$\mathcal{F} \subseteq {}^{X}\mathbb{R} \qquad \widetilde{\mathcal{F}} = \mathcal{F} \cap {}^{X}[0,1]$$

a set X, $\mathcal{F}, \mathcal{G} \subseteq {}^X\mathbb{R}$, $0 \in \mathcal{F}, \mathcal{G}$

We say that X has **a property** $\mathrm{wED}(\mathcal{F}, \mathcal{G})$, if

(1) for any sequence $\langle f_m : m \in \omega \rangle$ of functions from $\mathcal{F}$ converging to 0,

(2) there are sequences $\langle g_m : m \in \omega \rangle$ and $\langle h_m : m \in \omega \rangle$ of functions from $\mathcal{G}$ converging to 0 and

(3) there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$

such that for any $x \in X$ we have

$$h_m(x) \leq f_{n_m}(x) \leq g_m(x) \text{ for all but finitely many } m \in \omega.$$

$\{\min\{f, 1\};\ f \in \mathcal{G}\} \subseteq \mathcal{G}$

$\{\max\{f, 0\};\ f \in \mathcal{G}\} \subseteq \mathcal{G}$

X has $\mathrm{wED}(\widetilde{\mathcal{F}}, \mathcal{G})$ if and only if

(1) for any sequence $\langle f_m : m \in \omega \rangle$ of functions from $\widetilde{\mathcal{F}}$ converging to 0,

(2) there is a sequence $\langle g_m : m \in \omega \rangle$ of functions from $\widetilde{\mathcal{G}}$ converging to 0 and

(3) there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$

such that for any $x \in X$ we have

$$f_{n_m}(x) \leq g_m(x) \text{ for all but finitely many } m \in \omega.$$

Convergence of $\langle f_n : n \in \omega \rangle$, $\quad f_n, f : X \to \mathbb{R}$

Pointwise convergence P $\qquad f_n \xrightarrow{\text{P}} f$

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon)$$

Quasi-normal convergence QN $\qquad f_n \xrightarrow{\text{QN}} f$
there exists $\langle \varepsilon_n : n \in \omega \rangle$ converging to 0 such that

$$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to |f_n(x) - f(x)| < \varepsilon_n)$$

Discrete convergence D $\qquad f_n \xrightarrow{\text{D}} f$

$$(\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to f_n(x) = f(x))$$

L. Bukovský, I. Recław and M. Repický [2001] w$\mathcal{F}$QN-space

L. Bukovský and J. Š. [2013] wQN$_{\mathcal{F}}$-space

Let $\mathcal{F}$ be a family of functions on a set X. We say that X is a wQN$_{\mathcal{F}}$-space if each sequence of functions from $\mathcal{F}$ converging pointwise to zero on X has a subsequence converging quasi-normally.

$$\boxed{\mathrm{wQN}_{\mathcal{F}} = \mathrm{wED}(\mathcal{F}, \mathrm{Const})}$$

Lemma

Let X be a topological space, $\mathcal{F}, \mathcal{G}, \mathcal{H} \subseteq {}^X\mathbb{R}$. If X has $\mathrm{wED}(\mathcal{F}, \mathcal{G})$ *and* $\mathrm{wED}(\mathcal{G}, \mathcal{H})$ *then X has* $\mathrm{wED}(\mathcal{F}, \mathcal{H})$*.*

Lemma

Let X be a topological space, $\mathcal{F}, \mathcal{G} \subseteq {}^X[0, 1]$, $\mathrm{Const} \subseteq \mathcal{G}$*. If X has* $\mathrm{wQN}_{\mathcal{F}}$ *then X has* $\mathrm{wED}(\mathcal{F}, \mathcal{G})$*.*

Theorem

Let X be a topological space, $\mathrm{Const} \subseteq \mathcal{G} \subseteq \mathcal{F} \subseteq {}^X[0, 1]$*. X has* $\mathrm{wED}(\mathcal{F}, \mathcal{G})$ *and* $\mathrm{wQN}_{\mathcal{G}}$ *if and only if X has* $\mathrm{wQN}_{\mathcal{F}}$*.*

$$\begin{array}{ccccccc} & & \gamma\text{-set} & & & & \\ & & \downarrow & & & & \\ \text{QN} & \longrightarrow & \mathrm{S}_1(\Gamma,\Gamma) & \longrightarrow & \text{wQN} & \longrightarrow & \mathcal{U}_{fin}(\mathcal{O}_\omega,\Gamma) \\ \downarrow & & & & & & \\ \sigma\text{-set} & & & & & & \end{array}$$

due to L. Bukovský, I. Recław and M. Repický [1991], M. Scheepers [1996], [1997]
I. Recław [1997], J. Haleš [2005], L. Bukovský and J. Haleš [2007] and M. Sakai [2007]

L. Bukovský, I. Recław and M. Repický [1991]

A topological space X is a QN-space (a wQN-space) if each sequence of continuous real-valued functions converging to zero on X is (has a subsequence) converging quasi-normally.

$$\boxed{\mathrm{wQN} = \mathrm{wQN}_{\mathrm{C}(X)} = \mathrm{wED}(\mathrm{C}(X), \mathrm{Const})}$$

M. Scheepers [1996]

A topological space X is an $\mathrm{S}_1(\Gamma,\Gamma)$-space if for every sequence $\langle \mathcal{A}_n : n \in \omega\rangle$ of open γ-covers of X there exist sets $U_n \in \mathcal{A}_n, n \in \omega$ such that $\{U_n;\ n \in \omega\}$ is a γ-cover.

An infinite cover $\mathcal{A}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{A}$.
A topological space X is a σ-set if every F_σ subset of X is a G_δ set in X. (≤1933)

A topological space X is a γ-set if any open ω-cover of X contains γ-subcover.

Corollary (of Tsaban – Zdomskyy Theorem [2012])

If X is a perfectly normal space, $\mathcal{F} \subseteq \mathcal{B}$ and $\mathrm{Const} \subseteq \mathcal{G}$ then

$$\mathrm{QN} = \mathrm{wED}(\mathcal{B}, \mathrm{Const}) \to \mathrm{wED}(\mathcal{F}, \mathcal{G}).$$

Scheepers' Conjecture [1999]

Any perfectly normal wQN-space is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.

$$\begin{array}{ccccc} \mathsf{QN} & & \mathsf{S}_1(\Gamma,\Gamma) & & \\ \parallel\!\mid & & \parallel\!\mid & & \\ \mathsf{wQN}_{\tilde{\mathcal{L}}} & \rightarrow & \mathsf{wQN}_{\tilde{\mathcal{U}}} & \rightarrow & \mathsf{wQN} \end{array}$$

L. Bukovský [2008]

(1) Any $\mathrm{wQN}_{\tilde{\mathcal{L}}}$-space is a QN-space.

(2) Any $\mathrm{S}_1(\Gamma,\Gamma)$-space is a $\mathrm{wQN}_{\tilde{\mathcal{U}}}$-space.

B. Tsaban – L. Zdomskyy [2012]

Any perfectly normal QN-space is a $\mathrm{wQN}_{\tilde{\mathcal{L}}}$-space.

M. Sakai [2009]

Any $\mathrm{wQN}_{\tilde{\mathcal{U}}}$-space is an $\mathrm{S}_1(\Gamma,\Gamma)$-space.

$\mathrm{USC}_s \to \mathrm{wED}(\widetilde{\mathcal{U}}, \mathrm{C}(X))$

$$\begin{array}{ccccc} \text{QN} & & \mathrm{S}_1(\Gamma,\Gamma) & & \\ ||| & & ||| & & \\ \text{wQN}_{\widetilde{\mathcal{L}}} & \to & \text{wQN}_{\widetilde{\mathcal{U}}} & \to & \text{wQN} \end{array}$$

Theorem (H. Ohta – M. Sakai [2009])

Any wQN*-space with* USC_s *is an* $\mathrm{S}_1(\Gamma,\Gamma)$*-space.*

J. Haleš [2005], M. Sakai [2007], L. Bukovský and J. Haleš [2007]

Theorem (H. Ohta – M. Sakai)

Let X *be a perfectly normal space with* $\mathrm{Ind}(\mathrm{X}) = 0$.

(1)	X *possesses* USC.	$(1)^s$	X *possesses* USC_s.
(2)	X *is* (γ,γ)*-shrinkable.*	$(2)^s$	*Open* γ*-cover of* X *is shrinkable.*
(3)	X *is a* σ*-set.*	$(3)^s$	X *is a* $\gamma\gamma_{co}$*-space.*

Theorem

A topological space X *is an* $\mathrm{S}_1(\Gamma,\Gamma)$*-space if and only if* X *is a* wQN*-space with the property* $\mathrm{wED}(\widetilde{\mathcal{U}}, \mathrm{C}(X))$.

A topological space X is a σ-set if every F_σ subset of X is a G_δ set in X.

Any wQN-space with LSC_s is a $\mathrm{wQN}_{\widetilde{\mathcal{L}}}$-space.

Theorem (H. Ohta – M. Sakai [2009])

For a Tychonoff space X the following are equivalent.

(1) X *possesses* LSC.

(2) X *possesses* LSC_s.

(3) X *possesses* LSC_m.

(4) X *is a* P*-space and* . . . .

$\mathrm{LSC}_s \rightarrow \mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X))$

The only examples of perfectly normal space with LSC are all discrete spaces.

Theorem

(a) *A topological space X is a* $\mathrm{wQN}_{\widetilde{\mathcal{L}}}$*-space if and only if X is a* wQN*-space with the property* $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X))$.

(b) *A normal space X is a* $\mathrm{wQN}_{\widetilde{\mathcal{L}}}$*-space if and only if X is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space with the property* $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathcal{U})$.

(c) *A perfectly normal space X is a* QN*-space if and only if X has the Hurewicz property as well as the property* $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X))$.

Theorem

Let X be a perfectly normal space.

(1) X *has* $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X))$ *if and only if* X *has* $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathcal{U})$.

(2) *If* $\widetilde{\mathcal{B}}_1 \subseteq \mathcal{F} \subseteq \widetilde{\mathcal{B}}$ *then*

$$\mathrm{wED}(\mathcal{B}, \mathrm{C}(X)) \equiv \mathrm{wED}(\mathcal{F}, \mathrm{C}(X)) \equiv \mathrm{wED}(\mathcal{F}, \mathcal{U}).$$

Corollary

Let X be a perfectly normal space with Hurewicz property, $\widetilde{\mathcal{L}} \subseteq \mathcal{F} \subseteq \widetilde{\mathcal{B}}$. Then

$$\mathrm{QN} \equiv \mathrm{wED}(\mathcal{F}, \mathrm{C}(X)) \equiv \mathrm{wED}(\mathcal{F}, \mathcal{U}).$$

We say that a topological space X possesses Hurewicz property if for any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of countable open covers not containing a finite subcover there are finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$, $n \in \omega$ such that $\{\bigcup \mathcal{V}_n ;\ n \in \omega\}$ is a γ-cover.

a set X, $\mathcal{F}, \mathcal{G}, \mathcal{H} \subseteq {}^X\mathbb{R}$, $0 \in \mathcal{F}, \mathcal{G}, \mathcal{H}$

We say that X has **a property** $\mathrm{wED}^{\mathcal{H}}(\mathcal{F}, \mathcal{G})$, if

(1) for any sequence $\langle f_m : m \in \omega \rangle$ of functions from $\mathcal{F}$ converging to a function $f \in \mathcal{H}$,

(2) there are sequences $\langle g_m : m \in \omega \rangle$ and $\langle h_m : m \in \omega \rangle$ of functions from $\mathcal{G}$ converging to f and

(3) there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$

such that

$$h_m(x) \leq f_{n_m}(x) \leq g_m(x) \text{ for all but finitely many } m \in \omega.$$

Proposition

Let X be a perfectly normal space. The following are equivalent.

(1) *X possesses $\mathrm{wED}(\widetilde{\mathcal{U}}, \mathrm{C}(X))$.*

(2) *For any sequence $\langle f_m : m \in \omega \rangle$ of upper semicontinuous functions on X with values in $\mathbb{R}$ converging to F_σ-measurable function f, there is a sequence $\langle g_m : m \in \omega \rangle$ of continuous functions converging to f and an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$ such that $\langle f_{n_m} : m \in \omega \rangle \leq^* \langle g_m : m \in \omega \rangle$.*

Theorem

Let X be a perfectly normal space. Then

$$\mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X)) \equiv \mathrm{wED}(\mathcal{L}, \mathrm{C}(X)) \equiv \mathrm{wED}(\mathcal{U}, \mathrm{C}(X)) \equiv \mathrm{wED}^{\mathcal{B}}(\mathcal{L}, \mathrm{C}(X)) \equiv \mathrm{wED}^{\mathcal{B}}(\mathcal{U}, \mathrm{C}(X)).$$

a set X, $\mathcal{F}, \mathcal{G}, \mathcal{H} \subseteq {}^X\mathbb{R}$, $0 \in \mathcal{F}, \mathcal{G}, \mathcal{H}$

We say that X has **a property** $\mathrm{ED}^{\mathcal{H}}(\mathcal{F}, \mathcal{G})$, if

(1) for any sequence $\langle f_m : m \in \omega \rangle$ of functions from $\mathcal{F}$ converging to a function $f \in \mathcal{H}$,

(2) there are sequences $\langle g_m : m \in \omega \rangle$ and $\langle h_m : m \in \omega \rangle$ of functions from $\mathcal{G}$ converging to f and

~~(3) there is an increasing sequence of natural numbers $\{n_m\}_{m=0}^{\infty}$~~

such that

$$h_m(x) \leq f_m(x) \leq g_m(x) \text{ for all but finitely many } m \in \omega.$$

Theorem

Let X be a perfectly normal space. Then for any $\{0\} \subseteq \mathcal{F} \subseteq \mathcal{B}$ we have

$$\begin{aligned} \mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X)) &\equiv \mathrm{wED}^{\mathcal{F}}(\mathcal{L}, \mathrm{C}(X)) \equiv \mathrm{wED}^{\mathcal{F}}(\mathcal{U}, \mathrm{C}(X)) \\ &\equiv \mathrm{ED}^{\mathcal{F}}(\mathcal{L}, \mathrm{C}(X)) \equiv \mathrm{ED}^{\mathcal{F}}(\mathcal{U}, \mathrm{C}(X)). \end{aligned}$$

Theorem

(c) *A perfectly normal space X is a QN-space if and only if X has the Hurewicz property as well as the property $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathrm{C}(X))$.*

Theorem (L. Bukovský et al. [2001], B. Tsaban – L. Zdomskyy [2012])

A perfectly normal space X is a QN-space if and only if X has Hurewicz property and every F_σ-measurable function is discrete limit of continuous functions.

Theorem (Á. Császár – M. Laczkovich [1979], Z. Bukovská [1991])

Let X be a normal space, $f : X \to \mathbb{R}$. The following are equivalent.

(1) *f is a discrete limit of a sequence of continuous functions on X.*

(2) *f is a quasi-normal limit of a sequence of continuous functions on X.*

(3) *There is a sequence $\langle F_n : n \in \omega \rangle$ of closed subsets of X such that $f|F_n$ is continuous on F_n for any $n \in \omega$ and $X = \bigcup_{n\in\omega} F_n$.*

Discrete convergence $f_n \xrightarrow{\mathrm{D}} f \equiv (\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to f_n(x) = f(x))$

We say that a topological space X has **a property DL($\mathcal{F}, \mathcal{G}$)** if any function from $\mathcal{F}$ is a discrete limit of a sequence of functions from $\mathcal{G}$.

J. Cichoń – M. Morayne [1988], J. Cichoń – M. Morayne – J. Pawlikowski – S. Solecki [1991]

Theorem

(a) *Let X be a topological space. Then*

$$\mathrm{DL}(\mathcal{U}, \mathrm{C}(X)) \equiv \mathrm{DL}(\mathcal{L}, \mathrm{C}(X)) \equiv \mathrm{DL}(\widetilde{\mathcal{U}}, \mathrm{C}(X)) \equiv \mathrm{DL}(\widetilde{\mathcal{L}}, \mathrm{C}(X)) \equiv$$
$$(\forall Y \subseteq X)\ \mathrm{DL}(\mathcal{U}, \mathrm{C}(Y)) \equiv (\forall Y \subseteq X)\ \mathrm{DL}(\mathcal{L}, \mathrm{C}(Y)).$$

(b) *Let X be a separable metrizable space. Then*

$$\mathrm{DL}(\mathcal{U}, \mathcal{L}) \equiv \mathrm{DL}(\mathcal{L}, \mathcal{U}) \equiv \mathrm{DL}(\mathcal{L}, \mathrm{C}(X)) \equiv \mathrm{DL}(\mathcal{B}_1, \mathrm{C}(X)) \equiv \mathrm{DL}(\mathcal{B}, \mathrm{C}(X)).$$

Discrete convergence $\quad f_n \xrightarrow{\mathrm{D}} f \equiv (\forall x \in X)(\exists n_0)(\forall n \in \omega)(n \geq n_0 \to f_n(x) = f(x))$

Theorem

Let X be a perfectly normal space. If X has $\mathrm{wED}(\widetilde{\mathcal{L}}, \mathcal{U})$ then X has $\mathrm{DL}(\mathcal{B}_1, \mathrm{C}(X))$.

Proposition

If a topological space X has $\mathrm{DL}(\mathcal{U}, \mathcal{L})$ then X is a σ-set.

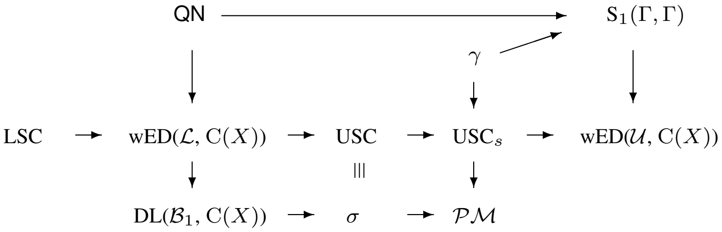
Corollary

Let $\widetilde{\mathcal{L}} \subseteq \mathcal{F} \subseteq {}^X\mathbb{R}$. Any perfectly normal space X possessing $\mathrm{wED}(\mathcal{F}, \mathcal{U})$ is a σ-set. Hence, X possesses $\mathrm{wED}(\mathcal{U}, \mathrm{C}(X))$.

Theorem (J.E. Jayne – C.A. Rogers [1982])

If A is an analytic subset of a Polish space then A has $\mathrm{DL}(\mathbf{\Delta}^0_2\text{-measurable}, \mathrm{C}(X))$.

Subsets of perfect Polish space



ZFC $\vdash$ wED($\mathcal{L}$, C(X)) $\nrightarrow$ QN

ZFC $\vdash$ wED($\mathcal{L}$, C(X)) $\nrightarrow$ LSC

ZFC $\vdash$ wED($\mathcal{U}$, C(X)) $\nrightarrow$ $S_1(\Gamma,\Gamma)$

ZFC $+\ \mathfrak{p} = \mathfrak{b} \vdash$ wED($\mathcal{U}$, C(X)) $\nrightarrow$ σ

ZFC $+\ \mathfrak{p} = \mathfrak{b} \vdash$ wED($\mathcal{U}$, C(X)) $\nrightarrow$ wED($\mathcal{L}$, C(X))

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Thanks for Your attention!