Luzinness on the real line

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Winter School, Hejnice 2010

Definition (Cardinal coefficients)

For any $I \subset \mathscr{P}(X)$ let

$$non(I) = min\{|A| : \; A \subset X \wedge A \notin I\}$$

$$add(I) = min\{|\mathscr{A}| : \; \mathscr{A} \subset I \wedge \bigcup \mathscr{A} \notin I\}$$

$$cov(I) = min\{|\mathscr{A}| : \; \mathscr{A} \subset I \wedge \bigcup \mathscr{A} = X\}$$

$$cov_h(I) = min\{|\mathscr{A}| : \; (\mathscr{A} \subset I) \wedge (\exists B \in Bor(X) \setminus I) \; (\bigcup \mathscr{A} = B)\}$$

$$cof(I) = min\{|\mathscr{A}| : \; \mathscr{A} \subset I \wedge \mathscr{A} - \text{Borel base of } I\}$$

$\mathbb{K}$ - σ ideal of meager sets
$\mathbb{L}$ - σ ideal of null sets

Definition

Let $I, J \subset \mathscr{P}(X)$ are σ - ideals on Polish space X with Borel base. We say that $L \subset X$ is a (I, J) - Luzin set if

- $L \notin I$
- $(\forall B \in I)\ B \cap L \in J$

If in addition the set L has cardinality κ then L is (κ, I, J) - Luzin set.

Definition

An ideals I and J are orthogonal in Polish space X if

$$\exists A \in \mathscr{P}(X)\ A \in I \wedge A^c \in J$$

and then we write $I \perp J$.

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Let $\mathscr{F} \subset X^X$ be any family of functions on the Polish space X. We say that $A, B \subset X$ are equivalent respect to $\mathscr{F}$ if

$$(\exists f, g \in \mathscr{F})\,(B = f[A] \wedge A = g[B])$$

Definition

We say that $A, B \subset X$ are Borel equivalent if A, B are equivalent respect to the family of all Borel functions.

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We say that σ - ideal I has Fubini property iff for every Borel set $A \subset X \times X$

$$\{x \in X : A_x \notin I\} \in I \Longrightarrow \{y \in X : A^y \notin I\} \in I$$

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Fact

Assume that $I \perp J$.

1. *There exist a (I, J) - Luzin set.*
2. *If L is a (I, J) - Luzin set then L is not (J, I) - Luzin set.*

Theorem (Bukovsky)

If κ is uncountable rebular cardinal and there are $(\kappa, \mathbb{K}, [\mathbb{R}]^{<\kappa})$ and $(\lambda, \mathbb{L}, [\mathbb{R}]^{<\lambda})$ - Luzin sets then

$$\kappa = cov(\mathbb{K}) = non(\mathbb{K}) = non(\mathbb{L}) = cov(\mathbb{L}) = \lambda.$$

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Theorem

Assume that $\kappa = cov(I) = cof(I) \leq non(J)$. *Let* $\mathcal{F}$ *be a family of functions from* $\mathscr{X}$ *to* $\mathscr{X}$. *Assume that* $|\mathcal{F}| \leq \kappa$. *Then we can find a sequence* $(L_\alpha)_{\alpha<\kappa}$ *such that*

1. L_α *is* (κ, I, J) *- Luzin set,*
2. *for* $\alpha \neq \beta$, L_α *is not equivalent to* L_β *with respect to the family* $\mathcal{F}$.

Remark

From proof of the above Theorem we have

$$(\forall \alpha, \beta, \zeta < \kappa)\ (\alpha \neq \beta \rightarrow |f_\zeta[L_\alpha] \setminus L_\beta| = \kappa).$$

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Let us notice that for every ideal I we have the inequality $cov(I) \leq cof(I)$. This gives the following corollary.

Corollary

If $2^\omega = cov(I) = non(J)$ then there exists continuum many different (I, J) - Luzin sets which aren't Borel equivalent. In particular, if CH holds then there exists continuum many different (ω_1, I, J) - Luzin sets which aren't Borel equivalent.

Corollary

If $2^\omega = cov(I) = non(J)$ then there exists continuum many different (I, J) - Luzin sets which aren't equivalent with respect to all I-measurable functions. In particular, if CH holds then there exists continuum many different (ω_1, I, J) - Luzin sets which aren't equivalent with respect to all I-measurable functions.

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Definiable (idealized) forcing was developed by J. Zapletal (see [8])

Lemma (folklore)

Let I be σ - ideal on 2^ω with conditions:

- $\mathbb{P}_I = Bor(2^\omega) \setminus I$ *be a proper,*
- I *has Fubini property.*

Assume that $B \in Bor(2^\omega) \cap I$ *be a Borel set in* $V[G]$*.*
Then there exists $D \in V$ *s.t.*

$$B \cap (2^\omega)^V \subset D \in I.$$

For Cohen and Solovay reals, see Solovay, Cichoń and Pawlikowski, see [2, 4, 7]

Proof

Let $\dot{B}$ – name for B
$\dot{r}$ – canonical name for generic real
then there exists $C \in Bor(2^\omega \times 2^\omega) \cap (I \otimes I)$ - Borel set coded in ground model V
$B = C_{\dot{r}_G}$ and $C \in I \otimes I$
Now by Fubini property:

$$\{x : \quad C^x \notin I\} \in I.$$

Let $x \in B \cap (2^\omega)^V$ then $V[G] \models x \in B$

$$0 < [\![x \in \dot{B}]\!] = [\![x \in C_{\dot{r}}]\!] = [\![(\dot{r}, x) \in C]\!] = [\![\dot{r} \in C^x]\!] = [C^x]_I$$

Then we have:

$$B \cap (2^\omega)^V \subset \{x : C^x \notin I\} \in I.$$

■

Definition

Let $M \subseteq N$ be standard transitive models of ZF.
Coding Borel sets from the ideal I is absolute iff

$$(\forall x \in M \cap \omega^\omega) M \vDash \#x \in I \leftrightarrow N \vDash \#x \in I.$$

Theorem

Let $\omega < \kappa$ *and* I, J *be* σ *- ideals with Borel base on* 2^ω*,*

- $\mathbb{P}_I = Bor(2^\omega) \setminus I$ *be a proper forcing notion,*
- I *has Fubini property,*
- *Borel codes for sets from ideal* J *are absolute.*

Then $\mathbb{P}_I = Bor(2^\omega) \setminus I$ *- is preserving* (I, J) *- Luzin set porperty.*

Proof

Let G is $\mathbb{P}_I$ generic over V

L - (κ, I, J) - Luzin set in the ground model V.

In $V[G]$ take any $B \in I$

then $L \cap B \cap V = L \cap B$ but by Lemma $L \cap B \in I$ in V

so we can find $b \in 2^\omega \cap V$ - Borel code s.t. $B \cap V \subset \#b \in I \cap V$

But L is (I, J)-Luzin set then $L \cap \#b \in J \cap V$,

Let $c \in 2^\omega \cap V$ be a Borel code s.t. $L \cap \#b \subset \#c \in J \cap V$ then by absolutness $\#c \in J$ in $V[G]$

finally we have in $V[G]$

$$L \cap B = L \cap B \cap V \subset L \cap \#b \subseteq \#c \in J \text{ in } V[G]. \qquad \blacksquare$$

Theorem

Let $(\mathbb{P}, \leq)$ *be a forcing notion such that*

$$\{B : B \in I \cap \mathrm{Borel}(\mathscr{X}), B \textit{ is coded in } V\}$$

is a base for I *in* $V^{\mathbb{P}}[G]$*. Assume that Borel codes for sets from ideals* I, J *are absolute. Then* $(\mathbb{P}, \leq)$ *preserve being* (I, J) *- Luzin sets.*

Corollary

Let $(\mathbb{P}, \leq)$ *be any forcing notion which does not change the reals i. e.* $(\omega^\omega)^V = (\omega^\omega)^{V^{\mathbb{P}}[G]}$. *Assume that Borel codes for sets from ideals* I, J *are absolute. Then* $(\mathbb{P}, \leq)$ *preserve being* (I, J) *- Luzin sets.*

Corollary

Assume that $(\mathbb{P}, \leq)$ *is a* σ*-closed forcing and Borel codes for sets from ideals* I, J *are absolute. Then* $(\mathbb{P}, \leq)$ *preserve* (I, J) *- Luzin sets.*

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Corollary

Let $\lambda \in On$ *be an ordinal number. Let* $\mathbb{P}_\lambda = \langle (P_\alpha, \dot{Q}_\alpha) : \ \alpha < \lambda \rangle$ *be iterating forcing with countable support. Spouse that*

1. *for any* $\alpha < \lambda$ $P_\alpha \Vdash \dot{Q}_\alpha - \sigma$ *closed ,*
2. *Borel codes for sets from ideals* I, J *are absolute,*

then $\mathbb{P}_\lambda$ *preserve* (I, J) *- Luzin sets.*

Measure case

Let Ω is a family of clopen sets of Cantor space 2^ω and

$$C^{random} = \{f \in \Omega^\omega : (\forall n \in \omega)\mu(f(n)) < 2^{-n}\}$$

with discrite topology.
Let us define $\sqsubseteq = \bigcup_{n\in\omega} \sqsubseteq_n$ where

$$(\forall f \in C^{random})(\forall g \in 2^\omega)(f \sqsubseteq_n g \leftrightarrow (\forall k \geq n)\ g \notin f(k)).$$

Definition (almost preserving)

We say that forcing notion P almost preserving relation $\sqsubseteq^{random}$ whenever for any countable elementary submodel $N \prec H_\kappa$ for enough large κ function g which covering $N \cap C^{random}$ with $P, \sqsubseteq^{random} \in N$ If $p \in P \cap N$ then there exists stronger condition $q \in P$ which is (N, P) generic s.t. $q \Vdash g$ covers $N[G]$

Definition of the notion of preservation of relation $\sqsubseteq^{random}$ by forcing notion $(\mathbb{P}, \leq)$ can be found in paper [5]. Let us focus on the following consequence of that definition.

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Theorem (Goldstern)

If $(\mathbb{P}, \leq)$ *preserves* $\sqsubseteq^{random}$ *then* $\mathbb{P} \Vdash \mu^*(2^\omega \cap V) = 1$.

Now we say that forcing notion $\mathbb{P}$ is preserving outer measure iff $\mathbb{P}$ preserve $\sqsubseteq^{random}$.

Theorem (Goldstern,Judah, Shelah)

Random forcing and Laver forcing preserves outer measure.

Here we cite from [5] the preservation Theorem:

Theorem (Goldstern)

Let $\mathbb{P}_\lambda = ((P_\alpha, Q_\alpha) : \alpha < \gamma)$ *be any countable support iteration such that*

$$(\forall \alpha < \gamma)\ P_\alpha \Vdash Q_\alpha \text{ preserves } \sqsubseteq^{random}$$

then $\mathbb{P}_\gamma$ *preserves the relation* $\sqsubseteq^{random}$*.*

Theorem

Assume that $\mathbb{P}$ *is a forcing notion which preserves* $\sqsubseteq^{random}$*. Then* $\mathbb{P}$ *preserves being* $(\mathbb{L}, \mathbb{K})$*-Luzin set.*

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Theorem

Assume that $\mathbb{P}$ is a forcing notion which preserves $\sqsubseteq^{random}$. Then $\mathbb{P}$ preserves being $(\mathbb{L}, \mathbb{K})$-Luzin set.

The analogous machinery can be used for ideal of meager sets $\mathbb{K}$. Here we rapidly recall from Goldstern paper [5] the necessary definitions. Let C^{Cohen} be set of all functions from $\omega^{<\omega}$ into itself. Then $\sqsubseteq^{Cohen} = \bigcup_{n\in\omega} \sqsubseteq_n^{Cohen}$ and for any $n \in \omega$ let

$$(\forall f \in C^{Cohen})(\forall g \in \omega^\omega)\ \ f \sqsubseteq_n^{Cohen} g \text{ iff}$$

$(\forall k < n)\ \ g \restriction k^\frown f(g \restriction k) \subseteq g$.
Then finally we have the following Theorem:

Theorem

Assume that $\mathbb{P}$ is a forcing notion which preserves $\sqsubseteq^{Cohen}$. Then $\mathbb{P}$ preserves being $(\mathbb{K}, \mathbb{L})$-Luzin set.

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