Some properties of $\mathcal{I}$-Luzin sets

joint work with Szymon Żeberski

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For each $A, B \subseteq \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ *and* $b \in \mathbb{R}$ *we define:*

$$\begin{aligned} A + B &= \{\bar{a} + \bar{b} : \ \bar{a} \in A, \bar{b} \in B\}, \\ \bar{x} + A &= \{\bar{x}\} + A, \\ bA &= \{b\bar{a} : \bar{a} \in A\}. \end{aligned}$$

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Furthermore for each $n \in \omega$ we denote:

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For a set $A \subseteq \mathbb{R}^n$ and $\bar{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $0 < k < n$, we define:

$$A_{\bar{x}} = \{(y_{k+1}, \ldots, y_n) : \ (x_1, \ldots, x_k, y_{k+1}, \ldots, y_n) \in A\}$$

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- *is translation invariant if for each $\bar{x} \in \mathbb{R}^n$ and $A \in \mathcal{I}$ we have $\bar{x} + A \in \mathcal{I}$;*

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- *$\mathcal{I}$-nonmeasurable if A doesn't belong to the σ-field $\sigma(\mathcal{B} \cup \mathcal{I})$ generated by Borel sets and the σ-ideal $\mathcal{I}$;*
- *completely $\mathcal{I}$-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for every $\mathcal{I}$-positive Borel set B.*

Definition

We say that a set A is an $\mathcal{I}$-Luzin set, if for each $I \in \mathcal{I}$ we have $|A \cap I| < |A|$.
A is called super $\mathcal{I}$-Luzin set, if A is an $\mathcal{I}$-Luzin set and for each $\mathcal{I}$-positive Borel set B we have $|A \cap B| = |A|$.

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For $\mathcal{M}$ and $\mathcal{N}$ σ-ideals of meager and null sets respectively we call a $\mathcal{M}$-Luzin set simply a Luzin set and a $\mathcal{N}$-Luzin set a Sierpiński set.

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Example

Let $\mathcal{I} = [\mathbb{R}^n]^{\leq\omega}$. Then a set A is $\mathcal{I}$-nonmeasurable iff it's not Borel and completely $\mathcal{I}$-nonmeasurable iff it's a Bernstein set. Furthermore all uncountable sets are $\mathcal{I}$-Luzin.

Definition

$\mathcal{I}$ has a Weaker Smital Property, if there exists a countable dense set D such that for each $\mathcal{I}$-positive Borel set A a set $A + D$ is $\mathcal{I}$-residual.
We say that the set D witnesses that $\mathcal{I}$ has the Weaker Smital Property.

The above notion was introduced in [**Bartoszewicz A., Filipczak M., Natkaniec T., On Smital Properties, 2011**].

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$\mathcal{I}$ has a Smital Property if $A + D$ is $\mathcal{I}$-residual for each $\mathcal{I}$-positive Borel set A and each dense set D.
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Proposition

Steinhaus Property $\Rightarrow$ Smital Property $\Rightarrow$ Weaker Smital Property.

Classic examples of σ-ideals that have all of the stated properties are $\mathcal{M}$ and $\mathcal{N}$. We may obtain further examples by Fubini products of ideals.

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Definition

Let $\mathcal{I} \subseteq P(\mathbb{R}^k)$ and $\mathcal{J} \subseteq P(\mathbb{R}^m)$ be σ-ideals. We define a σ-ideal $\mathcal{I} \otimes \mathcal{J} \subseteq P(\mathbb{R}^{k+m})$ as follows:

$$A \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow (\exists B \in \mathcal{B})(A \subseteq B \wedge \{\bar{x} \in \mathbb{R}^k : B_{\bar{x}} \notin \mathcal{J}\} \in \mathcal{I})$$

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Theorem (Bartoszewicz, Filipczak, Natkaniec, 2011)

If $\mathcal{I}$ and $\mathcal{J}$ have the Weaker Smital Property then $\mathcal{I} \otimes \mathcal{J}$ also has it.

Lemma

Let P and Q be disjoint perfect sets. Then there exist perfect sets $P' \subseteq P$ and $Q' \subseteq Q$ such that for each $x \in X$ a set $(x + P') \cap Q'$ contains at most one point.

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Remark (Grzegorz Plebanek, last week)

The above Lemma can be reformulated as follows:
For each Borel rectangle $P \times Q$ of uncountable sets exists Borel rectangle $P' \times Q' \subseteq P \times Q$ of uncountable sets such that a function $f(x, y) = x - y$ restricted to $P' \times Q'$ is an injection.

Theorem

There exists a translation invariant, containing uncountable sets σ-ideal $\mathcal{I}$ with Borel base for which there is an $\mathcal{I}$-measurable $\mathcal{I}$-Luzin set.

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Proof.

Let P' and Q' be perfect subsets from the previous Lemma for $P = [0,1] \times \mathbb{R}^{n-1}$ and $Q = [2,3] \times \mathbb{R}^{n-1}$. Set $\mathcal{I}$ to be a σ-ideal generated by translations of P' i.e.

$$\mathcal{I} = \{X \subseteq \mathbb{R}^n : \ (\exists C \in [\mathbb{R}^n]^\omega)(X \subseteq P' + C\}.$$

For each $I \in \mathcal{I}$ $Q' \cap I$ is countable, so Q' is an $\mathcal{I}$-Luzin set. $\square$

Declaration

From now on, we will assume that a σ-ideal $\mathcal{I}$ of subsets of $\mathbb{R}^n$

- is translation invariant,
- has a Borel base,
- has the Weaker Smital Property.

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Proof.

Let L be an $\mathcal{I}$-Luzin and suppose that it's not $\mathcal{I}$-nonmeasurable. Then there exists some $\mathcal{I}$-positive Borel set $B \subseteq L$ and we may find two disjoint perfect sets P and Q contained in B and furthermore, by Lemma, we may assume that for each $x \in X$ $|(P + x) \cap Q| \leq 1$.

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Let D witness the Weaker Smital Property. Then $P + D$ is $\mathcal{I}$-residual and $(P + D) \cap Q \notin \mathcal{I}$. On the other hand clearly $(P + D) \cap Q$ is countable. Contradiction completes the proof. □

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Corollary

Super $\mathcal{I}$-Luzin sets are completely $\mathcal{I}$-nonmeasurable.

Proposition

The existence of an $\mathcal{I}$-Luzin set implies the existence of an $\mathcal{I}$-Luzin set L such that $cf(|L|) > \omega$.

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Let L be an $\mathcal{I}$-Luzin set of cardinality $\mathfrak{c}$. Then there exists a linearly independent super $\mathcal{I}$-Luzin set.

Problem

Does the existence of an $\mathcal{I}$-Luzin set imply the existence of an $\mathcal{I}$-Luzin set which is a Hamel base?

Theorem

Let L be a linearly independent $\mathcal{I}$-Luzin set of cardinality $\mathfrak{c}$. Then there exists a set X such that $\{x + L : x \in X\}$ is a partition of $\mathbb{R}^n$.

Theorem (CH)

For each $\mathcal{I}$-Luzin set L there exists an $\mathcal{I}$-Luzin set X such that $\{x + L : x \in X\}$ is a partition of $\mathbb{R}^n$.

Assume in addition that $\mathcal{I}$ is scaling invariant i.e.

$$(\forall x \in \mathbb{R})(\forall A \in \mathcal{I})(xA \in \mathcal{I}).$$

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For each $n \in \omega\setminus\{0\}$ There exists an $\mathcal{I}$-Luzin set L such that $\bigoplus^n L$ is an $\mathcal{I}$-Luzin set and $\bigoplus^{n+1} L = \mathbb{R}^m$.

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Corollary (CH)

1. *There exists an $\mathcal{I}$-Luzin set L such that $\bigoplus^{n+1} L$ is an $\mathcal{I}$-Luzin for each $n \in \omega$,*
2. *There exists an $\mathcal{I}$-Luzin set L such that $L + L = L$,*
3. *There exists an $\mathcal{I}$-Luzin set L such that $\langle \bigoplus^{n+1} L : n \in \omega \rangle$ is an ascending sequence of $\mathcal{I}$-Luzin sets.*

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Theorem

It is consistent that $\mathfrak{c} = \omega_2$ and there is a Luzin set which is a linear subspace of $\mathbb{R}^n$.

Problem

Does the existence of a Luzin set imply the existence of a Luzin set which is a linear subspace of $\mathbb{R}^n$*?*

Theorem (CH)

There exists a Luzin set L such that $L + L$ is a Bernstein set.

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There exists a Sierpiński set S such that $S + S$ is a Bernstein set.

In [**Recław I., Some additive properties of special sets of reals, 1991**] author prooved that for every null set N and a perfect set P exists $P' \subseteq P$ such that N+P' remains null. Following lemmas generalize this result.

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Let A be a null set. We can find a perfect set P such that for every n

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Lemma

Let A be a meager set. We can find a perfect set P such that for every n

$$A + \bigoplus^{n} P \in \mathcal{M}.$$

Corollary

There exists a comeager null set R and perfect nowhere dense null set P such that $R + P \subseteq R$.

Theorem (Babinkostova, Sheepers, 2007)

Let L be a Luzin set such that for every $M \in \mathcal{M}$ $|L \cap M| \leq \omega$ and let S be a Sierpiński set such that for every $N \in \mathcal{N}$ $|L \cap M| \leq \omega$. Then $L + S$ is not a Bernstein set.

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Theorem

Assume that $\mathfrak{c}$ is a regular cardinal. There are no Luzin set L and Sierpiński set S such that $L + S$ is a Bernstein set.

Proof.

Regularity of $\mathfrak{c}$ implies that $|L| = |S| = \mathfrak{c}$. Let R and P be sets as in last Corollary. Let us denote $N = -R$ and $M = -N^c$. Then $P \subseteq (M + N)^c$. We will show that also $(L + S)^c$ also contains some perfect set.

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- $(L\cap A)+(S\cap B) \subseteq M+N$;
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- $(L\cap A)+(S\cap B)\subseteq M+N$;
- $(L\cap N)+(S\cap M^c)$ is a Luzin set;
- $(L\cap N^c)+(S\cap M)$ is a Sierpiński set;
- $|(L\cap N^c)+(S\cap M^c)| < \mathfrak{c}$.

Proof.

Regularity of $\mathfrak{c}$ implies that $|L| = |S| = \mathfrak{c}$. Let R and P be sets as in last Corollary. Let us denote $N = -R$ and $M = -N^c$. Then $P \subseteq (M+N)^c$. We will show that also $(L+S)^c$ also contains some perfect set.

$$\begin{aligned} L+S &= ((L\cap N)+(S\cap M)) \cup ((L\cap N)+(S\cap M^c)) \cup \\ &\cup ((L\cap N^c)+(S\cap M)) \cup ((L\cap N^c)+(S\cap M^c)) \end{aligned}$$

- $(L\cap A)+(S\cap B) \subseteq M+N$;
- $(L\cap N)+(S\cap M^c)$ is a Luzin set;
- $(L\cap N^c)+(S\cap M)$ is a Sierpiński set;
- $|(L\cap N^c)+(S\cap M^c)| < \mathfrak{c}$.

It follows that all of these sets have intersection with P of power lesser than $\mathfrak{c}$, so there exists perfect set $P' \subseteq P$ such that $P' \subseteq (L+S)^c$. Thus $L+S$ cannot be a Bernstein set.

Thank you for your attention!

Bibliography

- Babinkostova L., Sheepers M. Products and selection principles, Topology Proceedings, Vol. 31 (2007), 431-443.
- Bartoszewicz A., Filipczak M., Natkaniec T.,On Smital properties, Topology and its Applications (2011), Vol 158, 2066-2075.
- Michalski M., Żeberski Sz., Some properties of $\mathcal{I}$-Luzin sets (2015), Available at arXiv:1501.04900v1.
- Recław I., Some additive properties of special sets of reals, Colloquium Mathematicae, 62 (1991), 2, pp. 221-226.