Closure schemes on topological spaces

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Topologies generated by adding single points

Definition

We say that a topological space X is

- *Fréchet–Urysohn* if $\forall A \subseteq X$
  $\forall x \in \overline{A} \quad \exists S \subseteq A$ a sequence: $S \to x$,
- *radial* if $\forall A \subseteq X$
  $\forall x \in \overline{A} \quad \exists S \subseteq A$ a transfinite sequence: $S \to x$,
- *discretely Whyburn* if $\forall A \subseteq X$
  $\forall x \in \overline{A} \quad \exists D \subseteq A$ discrete: $\overline{D} \setminus A = \{x\}$,
- *Whyburn* if $\forall A \subseteq X$
  $\forall x \in \overline{A} \quad \exists B \subseteq A$: $\overline{B} \setminus A = \{x\}$.

Topologies generated by adding single points

Definition

We say that a topological space X is

- *sequential* if $\forall A \subseteq X$ non-closed
  $\exists x \in \overline{A} \setminus A \quad \exists S \subseteq A$ a sequence: $S \to x$,
- *pseudoradial* if $\forall A \subseteq X$ non-closed
  $\exists x \in \overline{A} \setminus A \quad \exists S \subseteq A$ a transfinite sequence: $S \to x$,
- *weakly discretely Whyburn* if $\forall A \subseteq X$ non-closed
  $\exists x \in \overline{A} \setminus A \quad \exists D \subseteq A$ discrete: $\overline{D} \setminus A = \{x\}$,
- *weakly Whyburn* if $\forall A \subseteq X$ non-closed
  $\exists x \in \overline{A} \setminus A \quad \exists B \subseteq A$: $\overline{B} \setminus A = \{x\}$.

Closure operators

Definition

We say that a mapping $C\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is a *closure operator* on a space X if the following holds:

- $\forall A, B \subseteq X\colon\ A \subseteq B \implies C(A) \subseteq C(B)$,
- $\forall A \subseteq X\colon\ A \subseteq C(A)$.

We also define additional properties of a closure operator C:

- $\forall A \subseteq X\colon\ C(C(A)) \subseteq C(A) \quad$ (*transitivity*),
- $\forall A, B \subseteq X\colon\ C(A \cup B) = C(A) \cup C(B) \quad$ (*additivity*),
- $C(\emptyset) = \emptyset \quad$ (*groundedness*).

Definition

Let C be a closure operator on X. A set $A \subseteq X$ is *C-closed* if $C(A) = A$, and it is *C-open* if its complement is C-closed.

Closure operators

Observation

- Intersection of any system of C-closed sets is a C-closed set. Hence, C-closed sets form a complete lattice. And dually for C-open sets.
- If we define $C_0(A) := A$, $C_{\alpha+1}(A) := C(C_\alpha(A))$, $C_\alpha(A) := \bigcup_{\beta<\alpha} C_\beta(A)$ for α limit, we get an increasing sequence of closure operators and $\overline{C}(A) := \bigcup_{\alpha\in\mathbf{On}} C_\alpha(A)$ is the transitive hull of the operator C.
- A set is C-closed if and only if it is $\overline{C}$-closed.

Closure schemes

Definition

A collection $\mathcal{C} = \langle C_X : X \in \textbf{Top} \rangle$ which assigns a closure operator C_X to each topological space X is called *closure scheme* if it holds that

- $\forall h \colon X \to Y$ homeo $\quad \forall A \subseteq X\colon \; h[C_X(A)] = C_Y(h[A])$,
- $\forall X \in \textbf{Top} \quad \forall A \subseteq X\colon \; C_X(A) \subseteq \overline{A}^X$.

Sometimes we write $\mathcal{C}(A, X)$ instead of $C_X(A)$.

The property of being $\mathcal{C}$-generated

Definition

Let $\mathcal{C}$ be a closure scheme, X a topological space.

- We say that the space X is *$\mathcal{C}$-generated* if $\forall A \subseteq X\colon\ C_X(A) = \overline{A}^X$, i.e. $C_X = \mathrm{cl}_X$.
- We say that the space X is *weakly* *$\mathcal{C}$-generated* if it is $\overline{\mathcal{C}}$-generated, i.e. $\forall A \subseteq X\colon\ \overline{C_X}(A) = \overline{A}^X$.
- We say that X is *$\mathcal{C}$-generated* *at a point* $x \in X$ if $\forall A \subseteq X\colon\ x \in \overline{A}^X \implies x \in C_X(A)$.
- We say that X is *weakly* *$\mathcal{C}$-generated* *at a point* $x \in X$ if it is $\overline{\mathcal{C}}$-generated at x, i.e. $\forall A \subseteq X\colon\ x \in \overline{A}^X \implies x \in \overline{C_X}(A)$.

Closure schemes – examples

Examples

- The topological closure $\mathrm{cl} = \langle \mathrm{cl}_X : X \in \mathbf{Top} \rangle$ is a transitive additive closure scheme. Every topological space is cl-generated.
- The scheme $\mathcal{C}^{\mathrm{Id}} = \langle \mathrm{id}_{\mathcal{P}(X)} : X \in \mathbf{Top} \rangle$ is a transitive additive closure scheme. A topological space is $\mathcal{C}^{\mathrm{Id}}$-generated iff it is discrete, and it is $\mathcal{C}^{\mathrm{Id}}$-generated at a point iff that point is isolated.
- We define closure schemes $\mathcal{C}^{\mathrm{Seq}}$, $\mathcal{C}^{\mathrm{Rad}}$, $\mathcal{C}^{\mathrm{D}}$, $\mathcal{C}^{\mathrm{DWh}}$, $\mathcal{C}^{\mathrm{Wh}}$ as

$$\begin{aligned}
\mathcal{C}^{\mathrm{Seq}}(A,X) &:= \{x \in X \ : \ \exists S \subseteq A \text{ a sequence}: \ S \to x\},\\
\mathcal{C}^{\mathrm{Rad}}(A,X) &:= \{x \in X \ : \ \exists S \subseteq A \text{ a transfinite sequence}: \ S \to x\},\\
\mathcal{C}^{\mathrm{D}}(A,X) &:= \bigcup\{\overline{D} \ : \ D \subseteq A \text{ discrete}\},\\
\mathcal{C}^{\mathrm{DWh}}(A,X) &:= A \cup \bigcup\{\overline{D} \ : \ D \subseteq A \text{ discrete}, |\overline{D} \setminus A| = 1\},\\
\mathcal{C}^{\mathrm{Wh}}(A,X) &:= A \cup \bigcup\{\overline{B} \ : \ B \subseteq A, |\overline{B} \setminus A| = 1\}.
\end{aligned}$$

$\mathcal{C}$-hereditary and $\mathcal{C}$-continuous mappings

Definition

Let $\mathcal{C}$ be a closure scheme, X and Y topological spaces. We say that a mapping $f\colon X \to Y$ is

- *$\mathcal{C}$-continuous* if
  $$\forall A \subseteq X\colon \; f[C_X(A)] \subseteq C_Y(f[A]);$$
- *$\mathcal{C}$-hereditary* if it is injective and
  $$\forall A \subseteq X\colon \; f[C_X(A)] \supseteq C_Y(f[A]) \cap \operatorname{rng}(f);$$
- *$\mathcal{C}$-continuous at a point* $x \in X$ if
  $$\forall A \subseteq X\colon \; x \in C_X(A) \implies f(x) \in C_Y(f[A]);$$
- *$\mathcal{C}$-hereditary at a point* $x \in X$ if it is injective and
  $$\forall A \subseteq X\colon \; x \in C_X(A) \impliedby f(x) \in C_Y(f[A]);$$

$\mathcal{C}$-hereditary and $\mathcal{C}$-continuous mappings

Proposition

- Continuity is the same thing as cl-continuity.
- If a mapping is $\mathcal{C}$-continuous, then the preimage of any $\mathcal{C}$-closed set is a $\mathcal{C}$-closed set. Moreover, if the scheme $\mathcal{C}$ is transitive, then the other implication also holds.

Proposition

Let $\mathcal{C}$ be a closure scheme, X, Y topological spaces, $f\colon X \to Y$.

- If f is $\mathcal{C}$-continuous, then it is $\overline{\mathcal{C}}$-continuous.
- If f is $\mathcal{C}$-hereditary, then it is $\overline{\mathcal{C}}$-hereditary if either
  - $\operatorname{rng}(f)$ is a $\mathcal{C}$-closed subset of Y or
  - $\operatorname{rng}(f)$ is a $\mathcal{C}$-open subset of Y and $\mathcal{C}$ is an additive scheme.

Preservation under subspaces, heredity

Theorem

Let $\mathcal{C}$ be a closure scheme, $x \in X$, a mapping $f\colon X \to Y$ $\mathcal{C}$-hereditary at x and continuous at x. If Y is $\mathcal{C}$-generated at $f(x)$, then X is $\mathcal{C}$-generated at x.

Theorem

Let $\mathcal{C}$ be a closure scheme and a mapping $f\colon X \to Y$ $\mathcal{C}$-hereditary and continuous.

- If Y is $\mathcal{C}$-generated, then X is also $\mathcal{C}$-generated.
- If Y is weakly $\mathcal{C}$-generated, then X is also weakly $\mathcal{C}$-generated if either
  - $\mathrm{rng}(f)$ is a $\mathcal{C}$-closed subset of Y or
  - $\mathrm{rng}(f)$ is a $\mathcal{C}$-open subset of Y and $\mathcal{C}$ is an additive scheme.

Preservation under subspaces, heredity

Corollary

Let $\mathcal{C}$ be a closure scheme.

- If all embeddings are $\mathcal{C}$-hereditary, then $\mathcal{C}$-generating is a hereditary property.
- If closed embeddings are $\mathcal{C}$-hereditary, then weak $\mathcal{C}$-generating is a closed hereditary property.
- If open embeddings are $\mathcal{C}$-hereditary and the scheme $\mathcal{C}$ is additive, then weak $\mathcal{C}$-generating is an open hereditary property.
- If all embeddings are both $\mathcal{C}$-hereditary and $\mathcal{C}$-continuous, then $\mathcal{C}$-generating coincides with hereditary weak $\mathcal{C}$-generating.

Preservation under subspaces, heredity – an example

Recall

We define a closure scheme $\mathcal{C}^{\mathsf{Seq}}$ as

$\mathcal{C}^{\mathsf{Seq}}(A, X) := \{x \in X \ : \ \exists S \subseteq X \text{ a sequence}: \ S \to x\}.$

Observation

- A topological space is $\mathcal{C}^{\mathsf{Seq}}$-generated iff it is Fréchet–Urysohn, and it is $\overline{\mathcal{C}^{\mathsf{Seq}}}$-generated iff it is sequential.
- $\mathcal{C}^{\mathsf{Seq}}$ is an additive closure scheme.
- All embeddings are both $\mathcal{C}^{\mathsf{Seq}}$-hereditary and $\mathcal{C}^{\mathsf{Seq}}$-continuous because convergence of sequences is absolute.
- A space is Fréchet–Urysohn iff it is hereditarily sequential.
- Fréchet–Urysohn spaces are closed under subspaces.
- Sequential spaces are closed under closed or open subspaces.

Preservation under inductive constructions

Definition

- A topology on X is *inductively generated by a family of mappings* $\{f_i \colon X_i \to X\}_{i \in I}$ if it is the finest topology such that all mappings f_i are continuous.
- A topology on X is *hereditarily inductively generated by a family of mappings* $\{f_i \colon X_i \to X\}_{i \in I}$ if the subspace topology of every $Y \subseteq X$ is inductively generated by the family $\{f_i \colon f_i^{-1}[Y] \to Y\}_{i \in I}$.

Examples

- Inductive generating: quotients, colimits.
- Hereditary inductive generating: hereditary quotients (in particular closed or open quotients), colimits with open colimit maps (in particular sums).

Preservation under inductive constructions

Theorem

Let $\mathcal{C}$ be a closure scheme, X a space inductively generated by a family of $\mathcal{C}$-continuous mappings $\{f_i \colon X_i \to X\}_{i \in I}$ such that all spaces X_i are $\mathcal{C}$-generated. Then the space X is $\mathcal{C}$-generated if at least one of the following conditions holds.

1. The closure scheme $\mathcal{C}$ is transitive.
2. X is hereditarily inductively generated by the family $\{f_i : i \in I\}$.

Corollary

Let $\mathcal{C}$ be a closure scheme, X a space inductively generated by a family of $\mathcal{C}$-continuous mappings $\{f_i \colon X_i \to X\}_{i \in I}$. If all spaces X_i are weakly $\mathcal{C}$-generated, then the space X is weakly $\mathcal{C}$-generated.

Preservation under inductive constructions

Corollary

Let $\mathcal{C}$ be a closure scheme.

- If embeddings of clopen subspaces are $\mathcal{C}$-continuous, then (weak) $\mathcal{C}$-generating is preserved under topological sums.
- If hereditarily (open, closed) quotient mappings are $\mathcal{C}$-continuous, then (weak) $\mathcal{C}$-generating is preserved under hereditary (open, closed) quotients.
- If quotient mappings are $\mathcal{C}$-continuous, then weak $\mathcal{C}$-generating is preserved under quotients.
- If continuous mappings are $\mathcal{C}$-continuous, then weak $\mathcal{C}$-generating is preserved under colimits.
- If open continuous mappings are $\mathcal{C}$-continuous, then $\mathcal{C}$-generating is preserved under colimits with open colimit maps.

Preservation under inductive constructions – an example

Observation

- A topological space is $\mathcal{C}^{\mathsf{Seq}}$-generated iff it is Fréchet–Urysohn, and it is $\overline{\mathcal{C}^{\mathsf{Seq}}}$-generated iff it is sequential.
- $\mathcal{C}^{\mathsf{Seq}}$ is an additive closure scheme.
- All embeddings are both $\mathcal{C}^{\mathsf{Seq}}$-hereditary and $\mathcal{C}^{\mathsf{Seq}}$-continuous because convergence of sequences is absolute.
- All continuous mappings are $\mathcal{C}^{\mathsf{Seq}}$-continuous because continuous mappings preserves convergence.
- A space is Fréchet–Urysohn iff it is hereditarily sequential.
- The class of Fréchet–Urysohn spaces is closed under subspaces, sums, and hereditary quotients.
- The class of sequential spaces is closed under closed or open subspaces, sums, quotients, and colimits.

Relations between the closure schemes

Definition

Let C^1, C^2 be closure operators on a set X. We define

$$C^1 \le C^2 \;:\Longleftrightarrow\; \forall A \subseteq X\colon\; C^1(A) \subseteq C^2(A).$$

Let $\mathcal{C}^1$, $\mathcal{C}^2$ be closure schemes. We define

$$\mathcal{C}^1 \le \mathcal{C}^2 \;:\Longleftrightarrow\; \forall X \in \mathbf{Top}\colon\; C^1_X \le C^2_X.$$

Observations

- "Closure schemes form a complete lattice." (But it is not even a proper class.)
- $\mathcal{C}^{\mathrm{Id}} \le \mathcal{C} \le \mathrm{cl}$ for any closure scheme $\mathcal{C}$.
- $\mathcal{C} \le \overline{\mathcal{C}}$ for any closure scheme $\mathcal{C}$.
- $\mathcal{C}^1 \le \mathcal{C}^2 \implies \overline{\mathcal{C}^1} \le \overline{\mathcal{C}^2}$ for any closure schemes $\mathcal{C}^1$, $\mathcal{C}^2$.

Relations between the closure schemes

Observation

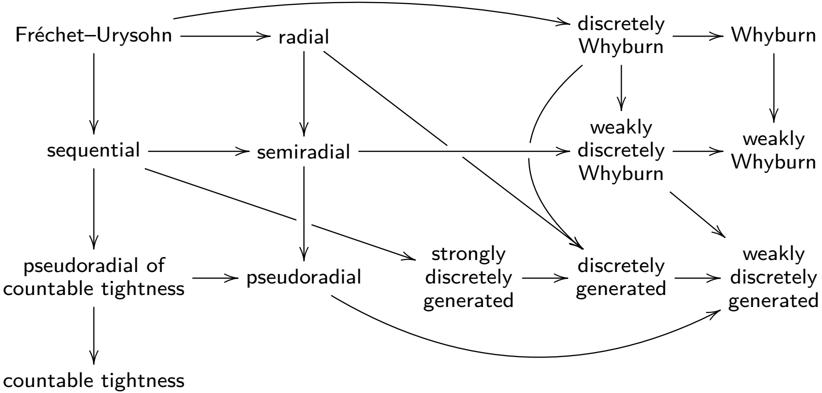
Let $\mathcal{C}^1$, $\mathcal{C}^2$ be closure schemes. Let us consider the following properties. It holds that $1 \implies 2 \implies 3$.

1. $\mathcal{C}^1 \leq \mathcal{C}^2$,
2. $\mathcal{C}^1$-generating at a point $\implies$ $\mathcal{C}^2$-generating at a point,
3. $\mathcal{C}^1$-generating $\implies$ $\mathcal{C}^2$-generating.

Proposition

Let $\mathcal{C}$ be a closure scheme, X a topological space, $x \in X$. If X is weakly $\mathcal{C}$-generated (at x) and Whyburn (at x), then it is $\mathcal{C}$-generated (at x).

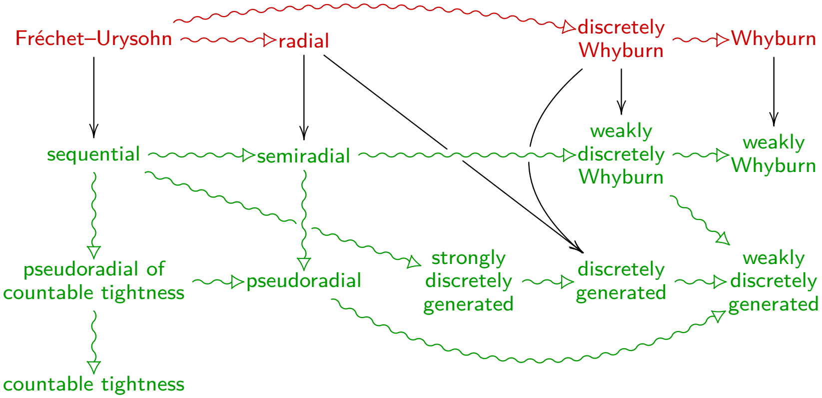
Relations between the closure schemes on T_2 spaces



Relations between the closure schemes on T_2 spaces

Example

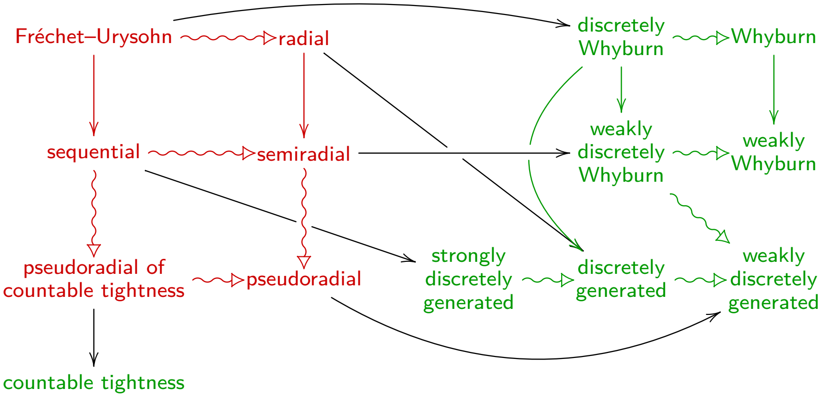
Arens' space is sequential but not Fréchet–Urysohn.



Relations between the closure schemes on T_2 spaces

Example

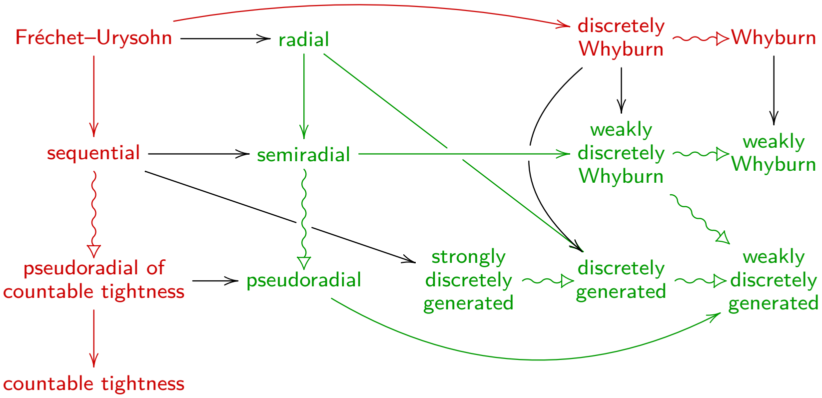
Reduced Arens' space is strongly discretely generated and discretely Whyburn because it contains only one non-isolated point. It has also countable tightness but it is not pseudoradial.



Relations between the closure schemes on T_2 spaces

Example

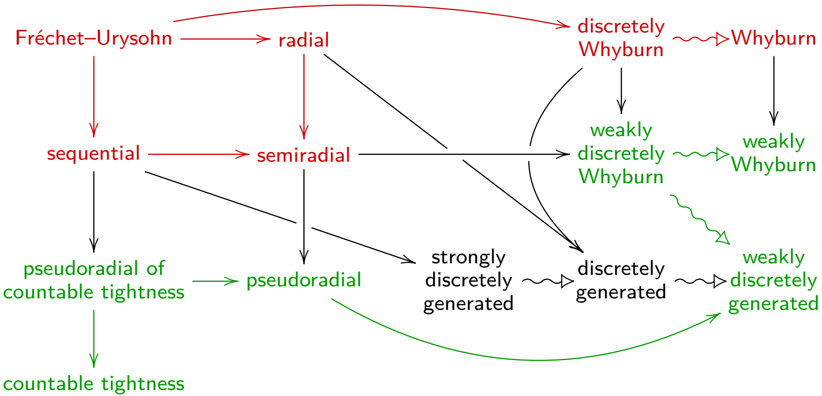
$(\omega_1 + 1)$ is radial and strongly discretely generated, but it is not Whyburn and it does not have countable tightness.



Relations between the closure schemes on T_2 spaces

Example

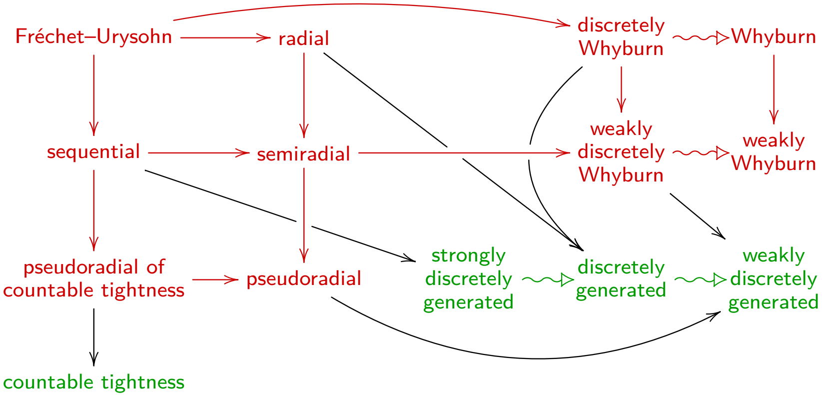
There exists a first countable, locally countable, uncountable T_3 space such that every uncountable subset contains a countable subset with uncountable closure (Simon, Tironi). After adjoining a point whose neighborhoods are complements of countable closed sets, we get a T_2 pseudoradial space of countable tightness which is not sequential.



Relations between the closure schemes on T_2 spaces

Example

If we refine the topology of $\beta\omega$ by $\{\omega \cup \{p\} : p \in \beta\omega \setminus \omega\}$, we obtain a strongly discretely generated space of countable tightness that is neither pseudoradial nor weakly Whyburn.



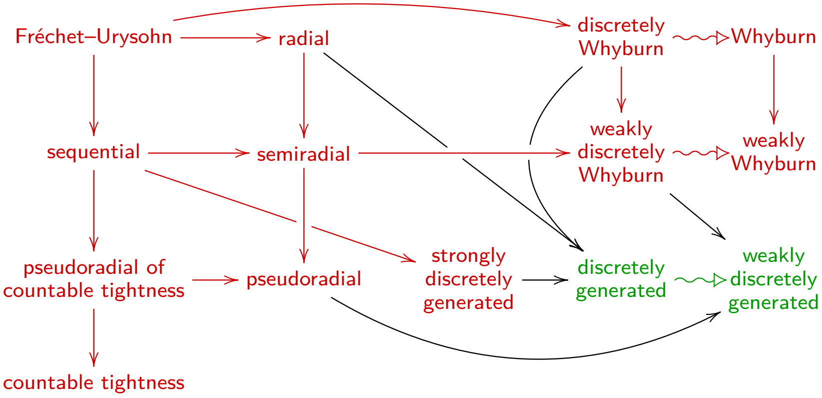
Relations between the closure schemes on T_2 spaces

Example

Let $p_0 \in \beta\omega \setminus \omega$. If we refine the topology of $\beta\omega$ by

$$\{\omega \cup \{p\} : p \in \beta\omega \setminus (\omega \cup \{p_0\})\} \cup \{\beta\omega \setminus A : A \subseteq \beta\omega \setminus \omega \text{ countable}\},$$

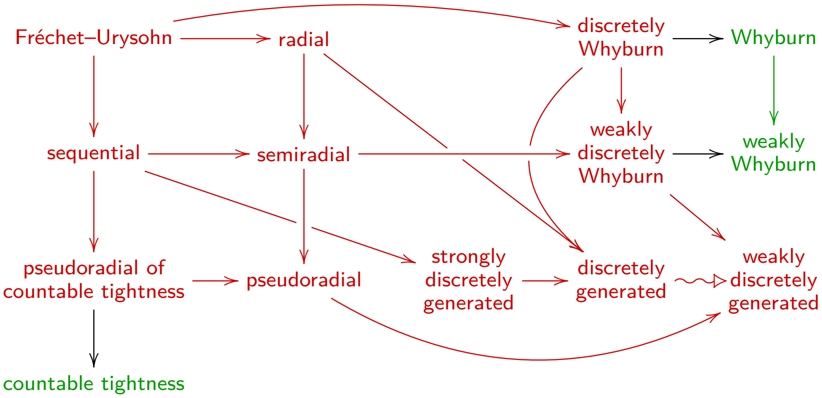
we obtain a discretely generated space that is not strongly discretely generated. It is also neither pseudoradial nor weakly Whyburn.



Relations between the closure schemes on T_2 spaces

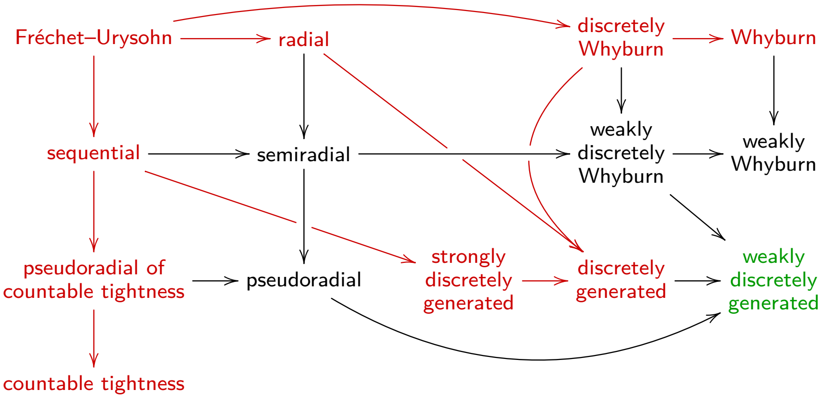
Example

Van Douwen's maximal space is Whyburn but not weakly discretely generated.



Example

Van Douwen's maximal space can be embedded into $2^{\mathfrak{c}}$. Hence, $2^{\mathfrak{c}}$ is weakly discretely generated but not discretely generated.

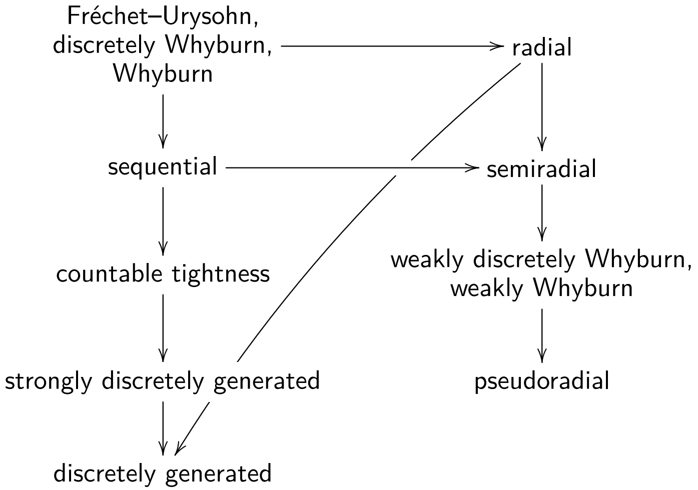


Relations between the schemes under additional conditions

Theorems

- Hausdorff locally compact spaces are weakly discretely generated. (Dow, Tkachenko, Tkachuk, Wilson)
- Hausdorff locally compact spaces of countable tightness are strongly discretely generated. (Dow, Tkachenko, Tkachuk, Wilson)
- Preregular locally compact weakly Whyburn spaces are pseudoradial. (Bella, Dow)
- Preregular locally countably compact Whyburn spaces are Fréchet–Urysohn. (Tkachuk, Yashchenko)

Relations between the schemes on T_2 compact spaces



Thank you for your attention.