Properness for iterations with uncountable supports

based on joint works of
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presented by AR

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Complete semi–purity

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λ is strongly inaccessible and D is a normal filter on λ

Definition 1

A forcing notion with λ–complete semi- purity is a triple $(\mathbb{Q}, \leq, \bar{\leq}_{\rm pr})$ such that $\bar{\leq}_{\rm pr} = \langle \leq^\alpha_{\rm pr} \colon \alpha < \lambda \rangle$ and $\leq, \leq^\alpha_{\rm pr}$ are transitive and reflexive (binary) relations on $\mathbb{Q}$ satisfying for each $\alpha < \lambda$:

(a) $\leq^\alpha_{\rm pr} \subseteq \leq$,

(b) $(\mathbb{Q}, \leq)$ is strategically $(<\lambda)$–complete and $(\mathbb{Q}, \leq^\alpha_{\rm pr})$ is strategically $(\leq\kappa)$–complete for all infinite cardinals $\kappa < \lambda$.

Note that unlike in Definition 17 of Part 2, in semi-purity we do not require any kind of pure decidability.

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Auxiliary Purity Game

Assume that $(\mathbb{Q}, \leq, \bar{\leq}_{\rm pr})$ is forcing notion with λ–complete semi-purity. Let $\bar{q} = \langle q_{\alpha,\eta} : \alpha < \lambda \ \& \ \eta \in {}^{\alpha}\alpha\rangle \subseteq \mathbb{Q}$ and let $p \in \mathbb{Q}$.

We define a game $\eth^{\rm aux}_{\lambda}(p, \bar{q}, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ between two players, COM and INC as follows. A play of $\eth^{\rm aux}_{\lambda}(p, \bar{q}, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ lasts λ steps during which the players choose successive terms of a sequence $\langle (r_\alpha, A_\alpha, \eta_\alpha, r'_\alpha) : \alpha < \lambda\rangle$ so that:

(a) $r_\alpha, r'_\alpha \in \mathbb{Q}$, $A_\alpha \in D$, $\eta_\alpha \in {}^{\alpha}\lambda$ and for $\alpha < \beta < \lambda$:

$$p = r_0 \leq r_\alpha \leq r'_\alpha \leq r_\beta \quad \text{and} \quad A_\beta \subseteq A_\alpha \quad \text{and} \quad \eta_\alpha \lhd \eta_\beta,$$

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At the end, COM wins the play $\langle (r_\alpha, A_\alpha, \eta_\alpha, r'_\alpha) : \alpha < \lambda \rangle$ if and only if both players had always legal moves (so the play really lasted λ steps) and

$(\odot)$ if $\gamma \in \mathop{\triangle}\limits_{\alpha<\lambda} A_\alpha$ is limit, then $\eta_\gamma \in {}^\gamma\gamma$ and $q_{\gamma,\eta_\gamma} \leq^\gamma_{\rm pr} r_\gamma$.

If COM has a winning strategy in $\Game^{\rm aux}_\lambda(p, \bar{q}, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ then we say that *the condition p is aux-generic over $\bar{q}$, D*.

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Main Purity Game

A game $\Game^{\rm main}_\lambda(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ between two players, Generic and Antigeneric, is defined as follows. A play of the game lasts λ steps during which the players construct a sequence $\langle \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda\rangle$. At stage $\alpha < \lambda$ of the play,

- first Generic chooses a system $\bar{p}^\alpha = \langle p_{\alpha,\eta} : \eta \in {}^\alpha\alpha\rangle$ of pairwise incompatible conditions from $\mathbb{Q}$.
- Then Antigeneric answers by picking a system $\bar{q}^\alpha = \langle q_{\alpha,\eta} : \eta \in {}^\alpha\alpha\rangle$ of conditions from $\mathbb{Q}$ satisfying

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($\boxdot$) there is an aux-generic condition $p^* \geq p$ over $\bar{q}, D$.

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Definition 2

A forcing notion $\mathbb{Q}$ is *λ–semi-purely proper over the filter D* if for some sequence $\bar{\leq}_{\rm pr}$ of binary relations on $\mathbb{Q}$,

- $(\mathbb{Q}, \leq, \bar{\leq}_{\rm pr})$ is a forcing with the λ–complete semi-purity and
- for every $p \in \mathbb{Q}$ Generic has a winning strategy in $\Game^{\rm main}_{\lambda}(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$.

Proposition 3

If a forcing notion $\mathbb{Q}$ is λ–semi-purely proper over D, then it is λ–proper in the standard sense.

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Proof of the Proposition

Let $\bar{\leq}_{\rm pr}$ be a sequence witnessing the semi-pure properness of $\mathbb{Q}$. Assume $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ satisfies

$$^{<\lambda}N \subseteq N, \quad |N| = \lambda \quad \text{and} \quad (\mathbb{Q}, \leq, \bar{\leq}_{\rm pr}), D \ldots \in N.$$

Let $p \in N \cap \mathbb{Q}$. Fix a winning strategy **st** $\in N$ of Generic in $\Game^{\rm main}_\lambda(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ and pick a list $\langle \underset{\sim}{\tau}_\alpha : \alpha < \lambda \rangle$ of all $\mathbb{Q}$–names for ordinals from N.

Consider a play of $\Game^{\rm main}_\lambda(p, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ in which Generic uses **st** and Antigeneric chooses his answers as follows. At stage $\alpha < \lambda$ of the play, after Generic played $\bar{p}^\alpha = \langle p_{\alpha,\eta} : \eta \in {}^\alpha\alpha \rangle$, Antigeneric picks $\bar{q}^\alpha = \langle q_{\alpha,\eta} : \eta \in {}^\alpha\alpha \rangle \in N$ such that for $\eta \in {}^\alpha\alpha$:

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We claim that p^* is $(N, \mathbb{Q})$–generic.

Suppose towards contradiction that $p^+ \geq p^*$, $p^+ \Vdash \underset{\sim}{\tau}_\beta = \zeta$, $\beta < \lambda$ but $\zeta \notin N$. Consider a play $\langle (r_\alpha, A_\alpha, \eta_\alpha, r'_\alpha) : \alpha < \lambda \rangle$ of $\Game^{\rm aux}_\lambda(p^*, \bar{q}, \mathbb{Q}, \leq, \bar{\leq}_{\rm pr}, D)$ in which COM follows her winning strategy and INC plays:

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Let $\gamma \in \underset{\alpha<\lambda}{\triangle} A_\alpha$ be a limit ordinal greater than β. Since the play was won by COM, we have $\eta_\gamma \in {}^\gamma\gamma$ and $q_{\gamma,\eta_\gamma} \leq^\gamma_{\rm pr} r_\gamma$. Since $p^+ \leq r_\gamma$, we know that $r_\gamma \Vdash \underset{\sim}{\tau}_\beta = \zeta$ and hence (by $(**)_{\eta_\gamma}$) $q_{\gamma,\eta_\gamma} \Vdash \underset{\sim}{\tau}_\beta = \zeta$. However, $q_{\gamma,\eta_\gamma} \in N$, contradicting $\zeta \notin N$.

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The iteration theorem

Theorem 4 ([RoSh:942, Thm 2.7])

Assume that λ is a strongly inaccessible cardinal and D is a normal filter on λ. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma \rangle$ be a λ–support iteration such that for every $\xi < \gamma$:

$$\Vdash_{\mathbb{P}_\xi} \text{“}\underset{\sim}{\mathbb{Q}}_\xi \text{ is } \lambda\text{–semi-purely proper over } D^{\mathbf{V}^{\mathbb{P}_\xi}}\text{”}$$

(where $D^{\mathbf{V}^{\mathbb{P}_\xi}}$ is the normal filter on λ generated in $\mathbf{V}^{\mathbb{P}_\xi}$ by D).

Then $\mathbb{P}_\gamma = \lim(\bar{\mathbb{Q}})$ is λ–proper in the standard sense.

Proof.

Somewhat like Proposition 3 plus trees of conditions plus stuff. . . □

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Proposition 5

The forcing notions $\mathbb{Q}^{\ell,\bar{E}}$ (for $\ell = 2,3,4$) and their bounded relatives $\mathbb{Q}^{\ell}_{\varphi,\bar{F}}$ (for $\ell = 2,3,4$) are λ–semi purely proper.

You may notice the absence of $\mathbb{Q}^{1,\bar{E}}_{E}$ and this may worry you if E is the club filter on λ (as this case was not covered by part 2). It is a strange case though.

Proposition 6 ([RoSh:942, Section 4])

Let E, E_t be club filters on λ.

1. *It is consistent that $\mathbb{Q}^{2,\bar{E}}$ is a dense subset of $\mathbb{Q}^{1,\bar{E}}_{E}$.*
2. *It is consistent that the complete Boolean algebras $\mathrm{RO}(\mathbb{Q}^{2,\bar{E}})$ and $\mathrm{RO}(\mathbb{Q}^{1,\bar{E}}_{E})$ are not isomorphic.*

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The forcing notions $\mathbb{Q}^{\ell,\bar{E}}$ *(for* $\ell = 2, 3, 4$*) and their bounded relatives* $\mathbb{Q}^{\ell}_{\varphi,\bar{F}}$ *(for* $\ell = 2, 3, 4$*) are* λ*–semi purely proper.*

You may notice the absence of $\mathbb{Q}^{1,\bar{E}}_{E}$ and this may worry you if E is the club filter on λ (as this case was not covered by part 2). It is a strange case though.

Proposition 6 ([RoSh:942, Section 4])

Let E, E_t *be club filters on* λ*.*

1. *It is consistent that* $\mathbb{Q}^{2,\bar{E}}$ *is a dense subset of* $\mathbb{Q}^{1,\bar{E}}_{E}$*.*
2. *It is consistent that the complete Boolean algebras* $\mathrm{RO}(\mathbb{Q}^{2,\bar{E}})$ *and* $\mathrm{RO}(\mathbb{Q}^{1,\bar{E}}_{E})$ *are not isomorphic.*

Proposition 5

The forcing notions $\mathbb{Q}^{\ell,\bar{E}}$ (for $\ell = 2, 3, 4$) and their bounded relatives $\mathbb{Q}^{\ell}_{\varphi,\bar{F}}$ (for $\ell = 2, 3, 4$) are λ–semi purely proper.

You may notice the absence of $\mathbb{Q}^{1,\bar{E}}_{E}$ and this may worry you if E is the club filter on λ (as this case was not covered by part 2). It is a strange case though.

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Let E, E_t be club filters on λ.

1. *It is consistent that $\mathbb{Q}^{2,\bar{E}}$ is a dense subset of $\mathbb{Q}^{1,\bar{E}}_{E}$.*
2. *It is consistent that the complete Boolean algebras $\mathrm{RO}(\mathbb{Q}^{2,\bar{E}})$ and $\mathrm{RO}(\mathbb{Q}^{1,\bar{E}}_{E})$ are not isomorphic.*

Diamonds are the best friends

For the rest of this talk we assume:

1. λ is a regular uncountable cardinal, $\lambda^{<\lambda} = \lambda$.
2. D is a normal filter on λ.
3. A set $\mathcal{S} \in D^+$ contains all successor ordinals below λ, $0 \notin \mathcal{S}$ and $\lambda \setminus \mathcal{S}$ is unbounded in λ. For an ordinal $\gamma < \lambda$ we set $\mathcal{S}[\gamma] = \mathcal{S} \setminus \{\delta \leq \gamma : \delta \text{ is limit }\}$.
4. $\mathcal{R}$ is the closure of $\lambda \setminus \mathcal{S}$ and $\bar{\gamma} = \langle \gamma_\alpha : \alpha < \lambda \rangle$ is the increasing enumeration of $\mathcal{R}$ (so the sequence $\bar{\gamma}$ is increasing continuous, $\gamma_0 = 0$ and all other terms of $\bar{\gamma}$ are limit ordinals).
5. There exists a $(D, \mathcal{S})$–diamond, where

Definition 7

A sequence $\bar{f} = \langle f_\delta : \delta \in \mathcal{S} \rangle$ is a $(D, \mathcal{S})$–*diamond* if $f_\delta \in {}^{\delta}\delta$ for $\delta \in \mathcal{S}$ and $(\forall \eta \in {}^{\lambda}\lambda)(\{\delta \in \mathcal{S} : f_\delta \vartriangleleft \eta\} \in D^+)$.

Another purity

Definition 8

Let $\mathbb{Q}$ be a forcing notion. A binary relation R^{pr} is called *a λ–sequential purity* on $\mathbb{Q}$ whenever $\bar{r}\ R^{\mathrm{pr}}\ r$ implies

(a) $\bar{r} = \langle r_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbb{Q}}$–increasing sequence of conditions from $\mathbb{Q}$ of limit length $\delta < \lambda$, and

(b) $r \in \mathbb{Q}$ is an upper bound of $\bar{r}$ (i.e., $r_\alpha \leq_{\mathbb{Q}} r$ for all $\alpha < \delta$).

If, additionally, the relation R^{pr} satisfies

(c) if $\bar{r} = \langle r_\alpha : \alpha < \delta \rangle\ R^{\mathrm{pr}}\ s_\beta$ for $\beta < \xi$, $\xi < |\delta|^+$ and $s_\beta \leq s_\gamma$ for $\beta < \gamma < \xi$,
then there is a condition $s \in \mathbb{Q}$ stronger than all s_β (for $\beta < \xi$) and such that $\bar{r}\ R^{\mathrm{pr}}\ s$,

then we say that R^{pr} is *a λ–sequential$^+$ purity* on $\mathbb{Q}$.

Candidates

Let $(\mathbb{Q}, \leq)$ be a strategically $({<}\lambda)$–complete forcing notion and $R^{\rm pr}$ be a λ–sequential purity on $\mathbb{Q}$.

Suppose that a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ and $\lambda, \mathbb{Q}, D, \mathcal{S}, \ldots \in N$

(but note we do not demand $R^{\rm pr} \in N$)

and a function $h : \lambda \longrightarrow N$ is such that its range ${\rm Rng}(h)$ includes $\mathbb{Q} \cap N$. Also, let $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle$ list all dense open subsets of $\mathbb{Q}$ belonging to N and let $\gamma < \lambda$.

Definition 9

1. We say that a sequence $\bar{f} = \langle f_\delta : \delta \in \mathcal{S} \rangle$ is *a* $(D, \mathcal{S}, h)$*–semi diamond for* $\mathbb{Q}$ *over* N if $f_\delta \in {}^\delta\delta$ for $\delta \in \mathcal{S}$ and
   - $(*)$ for every $\leq_{\mathbb{Q}}$–increasing sequence $\langle p_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{Q} \cap N$ we have that $\big\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h(f_\delta(\alpha)) = p_\alpha)\big\} \in D^+$.

   Below, let $\bar{f}$ be a $(D, \mathcal{S}, h)$–semi diamond for $\mathbb{Q}$ over N.
2. *An* $(N, h, \mathbb{Q}, R^{\mathrm{pr}}, \bar{f}, \bar{\mathcal{I}})$*–candidate* is a sequence $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ limit } \rangle$ of condition from $N \cap \mathbb{Q}$ satisfying for each limit $\delta \in \mathcal{S}$:
   - (a) if $h \circ f_\delta = \langle h(f_\delta(\alpha)) : \alpha < \delta \rangle \subseteq \mathbb{Q} \cap N$ and it has an upper bound in $\mathbb{Q}$, then $h(f_\delta(\alpha)) \leq q_\delta$ for all $\alpha < \delta$, and
   - (b) if, moreover, $h \circ f_\delta \in \mathrm{Dom}(R^{\mathrm{pr}})$, then also $h \circ f_\delta \; R^{\mathrm{pr}} \; q_\delta$, and
   - (c) if there is $q \in \bigcap_{\alpha<\delta} \mathcal{I}_\alpha$ such that $h \circ f_\delta \; R^{\mathrm{pr}} \; q$, then also $q_\delta \in \bigcap_{\alpha<\delta} \mathcal{I}_\alpha$.

The Diamond Game

Let $\bar{q} = \langle q_\delta : \delta \in \mathcal{S}\ \&\ \delta \text{ is limit } \rangle$ be a candidate and $r \in \mathbb{Q}$. We define a game $\Game^{\mathcal{S}}_\gamma(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$ of two players, *Generic* and *Antigeneric*, as follows. A play lasts $\leq \lambda$ moves and in the $i^{\rm th}$ move the players try to choose conditions $r^-_i, r_i \in \mathbb{Q}$ and a set $C_i \in D$ so that

(a) $r \leq r_i$, and $r^-_i \in N$, and if $i \notin \mathcal{S}[\gamma] \cap \mathcal{R}$ then $r^-_i \leq r_i$,

(b) $(\forall i < j < \lambda)(r_i \leq r_j\ \&\ r^-_i \leq r^-_j)$, and

(c) Generic chooses r^-_i, r_i, C_i if $i \in \mathcal{S}[\gamma]$, and Antigeneric chooses r^-_i, r_i, C_i if $i \notin \mathcal{S}[\gamma]$.

At the end Generic wins the play whenever both players always had legal moves (so the game lasted λ steps) and

($\circledast$) if $\delta \in \mathcal{S}[\gamma] \cap \bigcap\limits_{i<\delta} C_i$ is a limit ordinal and $h \circ f_\delta$ is an increasing sequence of conditions in $\mathbb{Q}$ such that for all $\alpha < \delta$ we have $h(f_\delta(\alpha + 1)) = r^-_{\alpha+1}$, then $q_\delta \leq r_\delta$ and $h \circ f_\delta\ R^{\rm pr}\ r_\delta$.

Definition 10

We say that a strategically $(<\lambda)$–complete forcing notion $\mathbb{Q}$ is *purely sequentially proper over* $(D, \mathcal{S})$*–semi diamonds* whenever the following condition $(\odot)$ is satisfied.

$(\odot)$ Assume that χ is a large enough regular cardinal and $N \prec \mathcal{H}(\chi)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ and $\lambda, \mathbb{Q}, D, \mathcal{S}, \ldots \in N$. **Then** there exists a λ–sequential purity $R^{\rm pr}$ on $\mathbb{Q}$ such that: for every ordinal $\gamma < \lambda$, a condition $p \in \mathbb{Q} \cap N$ and every $\bar{\mathcal{I}}, h, \bar{f}, \bar{q}$ satisfying

- $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle$ lists all open dense subsets of $\mathbb{Q}$ from N,
- a function $h : \lambda \longrightarrow N$ is such that $\mathbb{Q} \cap N \subseteq {\rm Rng}(h)$, and
- a sequence $\bar{f}$ is a $(D, \mathcal{S}, h)$–semi diamond for $\mathbb{Q}$, and
- $\bar{q}$ is an $(N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{\mathcal{I}})$–candidate,

we have that Generic has a winning strategy in the game $\Game^{\mathcal{S}}_{\gamma}(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$ for some condition $r \geq p$.

If the relation $R^{\rm pr}$ above can be required to be a λ–sequential$^+$ purity, then we say that $\mathbb{Q}$ is *purely sequentially$^+$ proper over* $(D, \mathcal{S})$*–semi diamonds*.

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we have that Generic has a winning strategy in the game $\Game^{\mathcal{S}}_{\gamma}(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$ for some condition $r \geq p$.

If the relation $R^{\rm pr}$ above can be required to be a λ–sequential$^+$ purity, then we say that $\mathbb{Q}$ is *purely sequentially$^+$ proper over* $(D, \mathcal{S})$*–semi diamonds*.

Proposition 11

If a forcing notion $\mathbb{Q}$ is purely sequentially proper over $(D, \mathcal{S})$–semi diamonds and there exists a $(D, \mathcal{S})$–diamond, then $\mathbb{Q}$ is λ–proper in the standard sense.

Proof Given N and $p \in \mathbb{Q} \cap N$.
Let $\bar{q}$ be an $(N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{\mathcal{I}})$–candidate and $r \geq p$ be such that Generic has a winning strategy in the game $\Game^{\mathcal{S}}_{\gamma}(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$.
Suppose that $\mathcal{I} \in N$ is an open dense subset of $\mathbb{Q}$, say $\mathcal{I} = \mathcal{I}_{j_0}$ (where $\bar{\mathcal{I}} = \langle \mathcal{I}_i : i < \lambda \rangle$ lists all open dense subsets of $\mathbb{Q}$ belonging to N). We want to argue that $\mathcal{I} \cap N$ is predense above r.
Suppose $r_0 \geq r$.

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Let $\bar{q}$ be an $(N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{\mathcal{I}})$–candidate and $r \geq p$ be such that Generic has a winning strategy in the game $\eth^{\mathcal{S}}_{\gamma}(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$.
Suppose that $\mathcal{I} \in N$ is an open dense subset of $\mathbb{Q}$, say $\mathcal{I} = \mathcal{I}_{j_0}$ (where $\bar{\mathcal{I}} = \langle \mathcal{I}_i : i < \lambda \rangle$ lists all open dense subsets of $\mathbb{Q}$ belonging to N). We want to argue that $\mathcal{I} \cap N$ is predense above r.
Suppose $r_0 \geq r$.

Consider a play of $\partial_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$ in which Generic follows her winning strategy and Antigeneric plays as follows.

- At stage $i = 0$, Antigeneric sets $C_0 = \lambda$, $r_0^- = \emptyset_{\mathbb{Q}}$ and r_0 is the one fixed above.
- At a stage $i \notin \mathcal{S}[\gamma]$, $i > 0$, Antigeneric first picks any legal move C_i, r_i^-, r_i' and then "corrects" it by choosing a condition $r_i \geq r_i'$ so that $r_i \in \bigcap_{j<i} \mathcal{I}_j$.

After the play is completed and a sequence $\langle C_i, r_i^-, r_i : i < \lambda \rangle$ is constructed, we know that Generic won, so:

($\circledast$) if $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i<\delta} C_i$ is a limit ordinal and $h \circ f_\delta$ is an increasing sequence of conditions in $\mathbb{Q}$ such that for all $\alpha < \delta$ we have $h(f_\delta(\alpha + 1)) = r_{\alpha+1}^-$, then $q_\delta \leq r_\delta$ and $h \circ f_\delta \; R^{\rm pr} \; r_\delta$.

Since $\bar{f}$ is a $(D, \mathcal{S}, h)$–semi diamond for $\mathbb{Q}$ over N, we know that

$$\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h(f_\delta(\alpha)) = r_\alpha^-)\} \in D^+.$$

Consider a play of $\Game^{\mathcal{S}}_\gamma(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$ in which Generic follows her winning strategy and Antigeneric plays as follows.

- At stage $i = 0$, Antigeneric sets $C_0 = \lambda$, $r_0^- = \emptyset_{\mathbb{Q}}$ and r_0 is the one fixed above.
- At a stage $i \notin \mathcal{S}[\gamma]$, $i > 0$, Antigeneric first picks any legal move C_i, r_i^-, r_i' and then "corrects" it by choosing a condition $r_i \geq r_i'$ so that $r_i \in \bigcap\limits_{j<i} \mathcal{I}_j$.

After the play is completed and a sequence $\langle C_i, r_i^-, r_i : i < \lambda\rangle$ is constructed, we know that Generic won, so:

($\circledast$) if $\delta \in \mathcal{S}[\gamma] \cap \bigcap\limits_{i<\delta} C_i$ is a limit ordinal and $h \circ f_\delta$ is an increasing sequence of conditions in $\mathbb{Q}$ such that for all $\alpha < \delta$ we have $h(f_\delta(\alpha+1)) = r^-_{\alpha+1}$, then $q_\delta \leq r_\delta$ and $h \circ f_\delta \ R^{\rm pr} \ r_\delta$.

Since $\bar{f}$ is a $(D, \mathcal{S}, h)$–semi diamond for $\mathbb{Q}$ over N, we know that

$$\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h(f_\delta(\alpha)) = r^-_\alpha)\} \in D^+.$$

Consider a play of $\eth^{\mathcal{S}}_{\gamma}(r, N, h, \mathbb{Q}, R^{\rm pr}, \bar{f}, \bar{q})$ in which Generic follows her winning strategy and Antigeneric plays as follows.

- At stage $i = 0$, Antigeneric sets $C_0 = \lambda$, $r^-_0 = \emptyset_{\mathbb{Q}}$ and r_0 is the one fixed above.
- At a stage $i \notin \mathcal{S}[\gamma]$, $i > 0$, Antigeneric first picks any legal move C_i, r^-_i, r'_i and then "corrects" it by choosing a condition $r_i \geq r'_i$ so that $r_i \in \bigcap\limits_{j<i} \mathcal{I}_j$.

After the play is completed and a sequence $\langle C_i, r^-_i, r_i : i < \lambda \rangle$ is constructed, we know that Generic won, so:

($\circledast$) if $\delta \in \mathcal{S}[\gamma] \cap \bigcap\limits_{i<\delta} C_i$ is a limit ordinal and $h \circ f_\delta$ is an increasing sequence of conditions in $\mathbb{Q}$ such that for all $\alpha < \delta$ we have $h(f_\delta(\alpha + 1)) = r^-_{\alpha+1}$, then $q_\delta \leq r_\delta$ and $h \circ f_\delta \ R^{\rm pr} \ r_\delta$.

Since $\bar{f}$ is a $(D, \mathcal{S}, h)$–semi diamond for $\mathbb{Q}$ over N, we know that

$$\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h(f_\delta(\alpha)) = r^-_\alpha)\} \in D^+.$$

Pick a limit ordinal $\delta \in \mathcal{S}[\gamma] \cap \underset{i<\lambda}{\triangle} C_i$ such that $\delta > j_0$, δ is a limit of elements of $\lambda \setminus \mathcal{S}$ and $h \circ f_\delta = \langle r_\alpha^- : \alpha < \delta \rangle$. Then by ($\circledast$) we have that $q_\delta \le r_\delta$ and $h \circ f_\delta \; R^{\mathrm{pr}} \; r_\delta$. Moreover, since $r_\alpha \le r_\delta$ for all $\alpha < \delta$ and since δ is a limit of points from $\lambda \setminus \mathcal{S}$ we get $r_\delta \in \bigcap_{j<\delta} \mathcal{I}_j$. Therefore $q_\delta \in \bigcap_{j<\delta} \mathcal{I}_j$, so in particular $q_\delta \in \mathcal{I}_{j_0} \cap N$. But the condition r_δ is stronger than q_δ and it is also stronger than r_0, so r_0 is compatible with q_δ.

Example 12

The following forcing notions are purely sequentially$^+$ proper over $(D, \mathcal{S})$–semi diamonds:

- $\leq \lambda$–strategically complete,
- $\mathbb{Q}^{\ell,\bar{E}}$ for $\ell = 2, 3, 4$,
- $\mathbb{Q}^{\ell}\varphi, \bar{F}$ for $\ell = 2, 3, 4$ (if λ is inaccessible),
- the forcing $\mathbb{Q}^*$ with $\mathcal{S}' = \lambda \setminus \mathcal{S}$ and many other.

The Iteration Theorem

Theorem 13 ([RoSh 1001, Thm 4.1])

Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathop{\mathbb{Q}}_{\sim}{}_\alpha : \alpha < \zeta^* \rangle$ *be a* λ*–support iteration such that for each* $\alpha < \zeta^*$

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\mathop{\mathbb{Q}}_{\sim}{}_\alpha \text{ is purely sequentially}^+ \text{ proper over } (D, \mathcal{S})\text{–semi diamonds ”.}$$

Then

1. $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ *is purely sequentially proper over* $(D, \mathcal{S})$*–semi diamonds.*
2. *If, additionally, for each* $\alpha < \zeta^*$

   $$\Vdash_{\mathbb{P}_\alpha} \text{“}\mathop{\mathbb{Q}}_{\sim}{}_\alpha \text{ is } ({<}\lambda)\text{–complete ”}$$

   then $\mathbb{P}_{\zeta^*}$ *is purely sequentially*$^+$ *proper over* $(D, \mathcal{S})$*–semi diamonds.*

A word about the proof

The proof of the theorem does not use trees of conditions at all (they are inconvenient for non-inaccessible case).

We play there games on more and more coordinates; at a crucial stage we use RS–conditions:

RS–condition in $\mathbb{P}_{\zeta^*}$ is a pair (p, w) such that $w \in [(\zeta^* + 1)]^{<\lambda}$ is a closed set, $0, \zeta^* \in w$, p is a function with domain $\mathrm{Dom}(p) \subseteq \zeta^*$, and

($\otimes$) for every two successive members $\varepsilon' < \varepsilon''$ of the set w, $p{\restriction}[\varepsilon', \varepsilon'')$ is a $\mathbb{P}_{\varepsilon'}$–name of an element of $\mathbb{P}_{\varepsilon''}$ whose support is included in the interval $[\varepsilon', \varepsilon'')$.

The end....

Thank you
for your attention during this tutorial!

References

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