Iterated forcing and the Continuum Hypothesis

Justin Tatch Moore

Cornell University

Winter School in Abstract Analysis
February 3,5,6 2015

This research and travel to this meeting was supported in part by grant DMS–1262019 from the US National Science Foundation.

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Theorem (Jensen)

Souslin's Hypothesis is consistent with CH.

Theorem (Eisworth-Roitman)

CH does not imply the existence of an Ostaszewski space: a perfectly normal countably compact noncompact space in which open sets are countable or co-countable.

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Theorem (Ishiu, M.)

Assume PFA^+. If L is a minimal non σ-scattered linear order, then L is either an A-line or a separable linear order of cardinality $\aleph_1$.

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- A new obstruction

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Part 1: basics and obstructions

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The second stage is typically where the challenge lies.

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Let $\langle S_n : n < \infty \rangle$ be a decreasing sequence of stationary sets with empty intersection. The iteration of the forcings Q_{S_n} adds closed unbounded sets $E_n \subseteq S_n$. In the ωth stage of the iteration, since $\bigcap_n E_n$ must be empty, it must be that ω_1 is collapsed (and consequently reals are added — e.g. a well ordering of ω in type ω_1).

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There is an ω-length iteration of forcings such that the iterands preserve stationary subsets of ω_1 and do not add reals but such that the iteration collapses ω_1.

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A countable support iteration of proper forcings is proper and in particular preserves ω_1.

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If Q is totally proper and q is (M, Q)-generic, it need not be true that q is totally (M, Q)-generic. It is true that any (M, Q)-generic condition in a totally proper forcing has a totally (M, Q)-generic extension.

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A ladder system (on ω_1) is a sequence C_α ($\alpha \in \lim(\omega_1)$) such that $C_\alpha \subseteq \alpha$ is cofinal and order type ω.

(U) For every ladder system $\mathbf{C}$ and $g : \omega_1 \to 2$, there is a $f : \omega_1 \to 2$ such that if $\delta \in \lim(\omega_1)$, then

$$f \restriction C_\delta \equiv^* g(\delta)$$

($f \equiv^* m$ means f constantly m with finitely many exceptions).

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Theorem (Devlin)

(U) implies $2^{\aleph_0} = 2^{\aleph_1}$.

Weak diamond and weak CH

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The negation of the statement in the previous theorem is known as weak diamond.

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The negation of the statement in the previous theorem is known as weak diamond. It represents the primary and best understood mechanism by which reals are introduced in an iteration of totally proper forcings.

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Proposition

If T is a pruned Aronszajn tree and some lexicographic order on T is minimal, then T is club minimal.

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Remark

The conjunction of (A) and $2^{\aleph_0} < 2^{\aleph_1}$ implies SH.

Part 2: completeness

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This is preceded by Shelah's notion of being $\mathbb{D}$-complete with respect to a simple completeness system, which we will refer to as complete properness. If P is totally proper and $\dot{Q}$ is a P-name for a completely proper forcing, then $P * \dot{Q}$ is complete.

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Observe: q_i forces that $\dot{h} \restriction C_\delta \equiv^* i$. Furthermore, if $H \subseteq M \cap P * \dot{Q}$ is M-generic, then q_i decides whether H/Γ_P has a lower bound.

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*The core theorem is due to Shelah. This is an amalgam of results of Shelah and Eisworth.

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As we will see, there are important examples of posets which are totally proper but not ω-proper. Posets which distinguish between higher levels of α-properness, however, tend to be artificial.

Examples

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The poset for force an instance of (U) is both ($< \omega_1$)-proper and remains proper in every outer model with the same reals. We have already illustrated how iterations of such posets can fail to be complete.

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Example

There is a poset which forces an instance of (A) which is completely proper and $(<\omega_1)$-proper.

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Let $\mathbf{C}$ be a ladder system and define $Q_{\mathbf{C}}$ to be the collection of all countable closed subsets of ω_1 which have finite intersection with every ladder in $\mathbf{C}$.

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Unless there is a club which is almost disjoint from $\mathbf{C}$, however, neither $Q_{\mathbf{C}}$ nor any other forcing adding such a club is ω-proper.

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Key Lemma

If M is a suitable model for $Q_{\mathbf{C}}$, C is a ladder in $M \cap \omega_1$, $D \subseteq Q_{\mathbf{C}}$ is a dense set in M, and $p \in Q_{\mathbf{C} \cap M}$, then there is a $q \leq p$ in $D \cap M$ such that $q \setminus p$ is disjoint from C.

Proof of Key Lemma for $Q_{\mathbf{C}}$

Proof.

Find an countable $N \prec H((2^{\aleph_0})^+)$ such that $N \in M$, and $p, D \in N$.

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Find an countable $N \prec H((2^{\aleph_0})^+)$ such that $N \in M$, and $p, D \in N$. Set $\alpha = \max(C \cap N)$ and define $q_0 = p \cup \{\alpha + 1\}$, noting that $q_0 \in N$.

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Proof of Key Lemma for Q_C

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Key Lemma implies completeness

Let $M \in N_0 \in N_1$ be suitable for $P * \dot{Q}_{\mathbf{C}}$, $G \subseteq P \cap M$ be M-generic, and $p * \dot{q} \in M$ with $p \in G$.

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Observe:

- if $\bar{p} \leq G$ is N_0 and N_1, generic, then $\bar{p}$ forces that $\dot{C}_\delta$ is in $\check{\mathscr{C}}$.

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Let $M \in N_0 \in N_1$ be suitable for $P * \dot{Q}_{\mathbf{C}}$, $G \subseteq P \cap M$ be M-generic, and $p * \dot{q} \in M$ with $p \in G$. Set $\delta = M \cap \omega_1$ and let $\mathscr{C}$ be the set of ladders in δ which are in N_0.

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$q \leq p$ if X_p is an initial part of X_q and $\mathscr{U}_p \subseteq \mathscr{U}_q$.

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Proposition (essentially Shelah)

Suppose that P is totally $(<\omega_1)$-proper and $\dot{T}$ is a P-name for a pruned A-tree. Then P forces $\dot{Q}_T$ is totally proper and $P * \dot{Q}_T$ is complete.

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If $D \in M$ is dense, $p \in Q_T \cap M$, and $\sigma \subseteq \omega^{M \cap \omega_1}$ is a finite set of virtual elements consistent with p,

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If $D \in M$ is dense, $p \in Q_T \cap M$, and $\sigma \subseteq \omega^{M \cap \omega_1}$ is a finite set of virtual elements consistent with p, then there is a $q \leq p$ in $D \cap M$ which is consistent with σ.

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Assume for simplicity $\sigma \in T^{[n]}$ for some n. Set $n = |\sigma|$ and let $f : \omega_1 \to T^{[n]}$ be in M such that $f(\delta) = \sigma$.

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$$U = \{f(\nu) \restriction \xi : \xi < \nu \text{ and } \nu \in A\}.$$

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and observe that U is a pruned subtree of $T^{[n]}$. If $p' = (x_p, \mathscr{U}_p \cup \{U\})$, then $p' \in Q_T \cap M$. Also if $q \leq p'$ is in $D \cap M$, then there is a $\nu > \alpha_q$ in A such that $f(\nu)$ extends a tuple from the last level of x_q, contradicting the definition of A. □

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Proposition (M.)

The poset $Q_{T,\mathbf{C},g}$ is completely proper and $(< \omega_1)$-proper.

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The actually situation is somewhat more complicated: we must deal with the fact T_δ and C_δ may only be determined up to a countable number of candidates. The basic idea remains the same.

Part 3: completeness is not enough

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1. $P_\xi * \dot{Q}_\xi$ *is complete for all* $\alpha \in \theta$ *and*
2. *either*
   - A *for all* $\xi \in \theta$, $\dot{Q}_\xi$ *is forced to be (weakly)* α*-proper for all* $\alpha \in \omega_1$ *or*
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Problem (Shelah)

Is the forcing axiom for completely proper forcings consistent with CH?

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Assume CH. There is a tree T of height ω_1 such that:

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In particular, the forcing axioms for completely proper forcings is not consistent with CH. Also, by joint work with Aspero and Larson, there are variations of the forcing axiom for completely proper forcings which are individually consistent with CH but which jointly imply $2^{\aleph_0} = 2^{\aleph_1}$.

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The construction starts by selecting a club E_0 in $L[\mathrm{ind}]$ such that T_{E_0} has no uncountable branch in $L[\mathrm{ind}]$.

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Item 4 implies that $T^2 \setminus \Delta$ is special.

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Assume CH. There is a Δ_0-definable mapping $(\mathrm{ind}, E) \mapsto T_E^{\mathrm{ind}}$ *satisfying the previous conditions such that if* T_E^{ind} *has no uncountable branch, then* T_E^{ind} *is completely proper.*

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Proof of Shelah's proposition

Suppose not. Define $\delta_{\alpha,\xi}$ by recursion for $\alpha \in \omega_1$ and $\xi \in \omega^2$.

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$E_{\xi+1} \cap \delta_{\alpha,0}$ is the unique element of $T_\xi \cap \mathscr{P}(\delta_{\alpha,0}+1)$ which contains $\delta_{\beta,\omega\cdot k}$ whenever $\beta \in \alpha$ and $\xi \in \omega \cdot k$.

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If $\xi > 0$ is a limit ordinal, then

$$E_\xi \cap \delta_{\alpha,0} = \bigcap\{E_\eta \cap \delta_{\alpha,0} : \eta \in \xi\}$$

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The proof of the claim is finished by noting

$$\delta_{\alpha,\omega\cdot k+i+1} = \mathrm{ind}(E_{\omega\cdot k+i+1} \cap \delta_{\alpha,0}).$$

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Note that given $E_\xi \cap \delta_{\alpha,0}$, we know $T_{E_\xi} \cap \mathscr{P}(\delta_{\alpha,0} + 1)$. This justifies the reference to T_ξ in the definition of $E_{\xi+1} \cap \delta_{\alpha,0}$.

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Here $\theta_0, \ldots, \theta_3$ are logical formulas whose truth is defined by recursion...

$\theta_0^\delta(x,t,\beta)$ $\max(t) \in \min(x)$, $t \cup x$ is in $T_E \cap \mathscr{P}(\beta)$, and

$$\mathrm{otp}(E \cap \min(x))^* = \mathrm{ind}(t, n)$$

for some $n \in \omega$;

$\theta_1^\delta(x,t,\beta)$ if D is a dense subset of $T_E \cap \mathscr{P}(\nu)$ for some limit ordinal $\nu \in \beta$, $\mathrm{ind}(D) \in \beta$, and

$$\mathrm{otp}(E \cap \min(x))^* = \mathrm{ind}(t, e_\delta(\mathrm{ind}(D))),$$

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$\theta_2^\delta(x,t,\beta)$ if $y \subseteq \beta$, $e_\delta(\min(y)) \in e_\delta(\min(x))$ and

$$\theta_0^\delta \wedge \theta_1^\delta \wedge \theta_2^\delta \wedge \theta_3^\delta(y,t,\beta),$$

then $x \cap y \subseteq \{\min(x)\}$.

$\theta_3^\delta(x,t,\beta)$ if $s, z \subseteq \beta$, $\min(z) = \min(x)$, and

$$\theta_0^\delta \wedge \theta_1^\delta \wedge \theta_2^\delta(z,s,\beta),$$

then $\mathrm{ind}(x) \leq \mathrm{ind}(z)$.

Open Problems and Concluding Remarks

The posets needed to prove:

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This would solve:

Problem

Is it consistent that whenever L is a non σ-scattered linear order then there an $L' \subseteq L$ which is non σ-scattered such that L does not embed into L'?

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Consider the following statement:

(μ) If $\langle D_\alpha : \alpha \in \omega_1 \rangle$ satisfies D_α is a closed subset of α for each $\alpha \in \omega_1$, then there is a club $E \subseteq \omega_1$ such that for all $\alpha \in \omega_1$ there is an $\bar{\alpha} \in \alpha$ with:

$$E \cap (\bar{\alpha}, \alpha) \subseteq D_\alpha \qquad \text{or} \qquad E \cap (\bar{\alpha}, \alpha) \cap D_\alpha = \emptyset$$

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An instance of (μ) can be forced with a completely proper poset.

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(R) If $\langle D_\alpha : \alpha \in \omega_1 \rangle$ satisfies $D_\alpha \subseteq \alpha$ has ordertype less than α for all limit ordinals α, then there is a club E satisfying the conclusion of (μ).

(D) The map $\xi \mapsto \mathrm{ind}(\dot{g} \restriction \xi)$ is forced to be $\leq_{\mathrm{NS}}$-dominating, where $\dot{g}$ is the name for the generic element of 2^{ω_1} with respect to the poset $2^{<\omega_1}$.

Open Problems and Concluding Remarks

Proposition (essentially Shelah)

(R) is consistent with CH; it can be forced by iterating forcings which are absolutely totally proper.

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Problem (M.)

Is the conjunction of (D) and (R) consistent with CH?

Thank you for your attention!