Topological applications of long ω_1-approximation sequences II

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Let P be a poset. For $p \in P$, let $p\!\uparrow = \{q : q \geq p\}$.

Definition (Peregudov). *Define the Noetherian type* Nt(P) *of* P *to be the least infinite cardinal* κ *for which* $|p\!\uparrow| < \kappa$ *for all* $p \in P$.

Define the Noetherian type Nt(X) *of a topological space* X *to be the least* Nt($\mathcal{B}$) *where* $\mathcal{B}$ *is a base of* X *and* $\mathcal{B}$ *is ordered with respect to* $\subset$.

(Recall that a topological base is a family $\mathcal{B}$ of open sets such that for every $p \in U$ with U open, some $B \in \mathcal{B}$ satisfies $p \in B \subset U$.)

As a topological cardinal function, Nt is somewhat unusual. A few examples:

• If $\mathcal{B}$ is a base of X, then $\mathrm{Nt}(X^{|\mathcal{B}|}) = \aleph_0$. Hence, there are compact spaces X, Y such that $\mathrm{Nt}(X \times Y) < \max\{\mathrm{Nt}(X), \mathrm{Nt}(Y)\}$.

• There are Tychonoff spaces X, Y such that

$$\mathrm{Nt}(X \times Y) < \min\{\mathrm{Nt}(X), \mathrm{Nt}(Y)\}.$$

We do not know if there is a compact example of this. However, GCH implies that $\mathrm{Nt}(X^n) = \mathrm{Nt}(X)$ for all compact homogeneous X.

• The countably supported box product topology on $2^{\aleph_\omega}$ has Noetherian type in $[\aleph_1, \aleph_4]$, with $\aleph_1$ and $\aleph_2$ consistent, and the consistency of $\aleph_3$ and $\aleph_4$ unknown.

References

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D. Milovich, *Noetherian types of homogeneous compacta and dyadic compacta*, *Topology and its Applications* **156** (2008), 443–464.

S. A. Peregudov, *On the Noetherian type of topological spaces*, Comment. Math. Univ. Carolin. **38** (1997), no. 3, 581–586.

A compact space is *dyadic* if it is a continuous image of some 2^κ.

If X is the quotient of $2^\omega \oplus 2^{\omega_1}$ induced by identifying $(0)_{n<\omega}$ and $(0)_{\alpha<\omega_1}$, then X is dyadic and $\mathrm{Nt}(X) = \aleph_2$.

More generally, $\mathrm{Nt}(X) > \kappa$ if κ is a regular cardinal, X is a space, $p \in X$, some local π-base at p is smaller than κ, and no local base at p is smaller than κ.

Recall that a local base (local π-base) at p is a coinitial family $\mathcal{U}$ of open neighborhoods of p. That is, $p \in U$ ($U \neq \varnothing$) and U is open for all $U \in \mathcal{U}$, and if $p \in O$ and O is open, then $U \subset O$ for some $U \in \mathcal{U}$.

A space H is *homogeneous* if for all $p, q \in H$ there exists a homeomorphism $f\colon H \to H$ such that $f(p) = q$.

Theorem (Milovich, 2008). $\mathrm{Nt}(X) = \aleph_0$ *for all homogeneous dyadic compact* X.

Corollary. $\mathrm{Nt}(G) = \aleph_0$ *for all compact groups* G.

Proof. All topological groups are homogeneous. By the Ivanovskiĭ–Kuz′minov Theorem (1959), compact groups are also dyadic. $\square$

The *weight* $\mathsf{w}(X)$ of a space X is the least infinite cardinal κ such that X has a base not larger than κ.

The π-*character* $\pi\chi(p, X)$ of a point p in a space X is the least infinite cardinal κ such that p has a local π-base not larger than κ.

Theorem (Gerlits, 1976; Efimov, 1977)**.** *If X is compact and dyadic, then* $\sup_{p\in X} \pi\chi(p, X) = \mathsf{w}(X)$*.*

Corollary. *If X is compact, homogeneous, and dyadic, then, for all $p \in X$, $\pi\chi(p, X) = \mathsf{w}(X)$.*

Theorem (Milovich–Spadaro, 2014)**.** *If X is compact, κ is a regular uncountable cardinal,* $\mathsf{w}(X) \geq \kappa$*, and $\pi\chi(p, X) < \kappa$ on a dense set of $p \in X$, then* $\mathsf{Nt}(X) > \kappa$*.*

Every metric space has Noetherian type $\aleph_0$. Why? Take $\mathcal{B} = \bigcup_{n<\omega} \mathcal{R}_n$ where each $\mathcal{R}_n$ is a locally finite open cover refining the balls of diameter 2^{-n}.

A topological base $\mathcal{B}$ is called *efficient* if

- it has Noetherian type $\aleph_0$,
- $U \subsetneq V \Rightarrow \overline{U} \subset V$ for all $U, V \in \mathcal{B}$, and
- for all infinite $\mathscr{S} \subset \mathcal{B}$, the set $\{T \in \mathcal{B} : \exists S \in \mathscr{S}\ \ S \subsetneq T\}$ is infinite.

Lemma. *Every base of a compact metric space K contains an efficient base of K.*

Proof. Given a base $\mathcal{B}$ of K, we will choose a sequence $(\mathcal{A}_n)_{n<\omega}$ of finite open subcovers of $\mathcal{B}$ such that $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$ will be an efficient base.

Given $n < \omega$ and $(\mathcal{A}_m)_{m<n}$, choose, for each $p \in K$, an neighborhood N_p of p in $\mathcal{B}$ sufficently small that

1. $\mathrm{diam}(N_p) \leq 2^{-n}$,
2. $\overline{N_p} \subset \bigcap\{A : p \in A \in \bigcup_{m<n} \mathcal{A}_m\}$,
3. $N_p \cap A = \varnothing$ or $N_p = A$ for all singleton $A \in \bigcup_{m<n} \mathcal{A}_m$, and
4. $\mathrm{diam}(N_p) < \mathrm{diam}(A)$ for all non-singleton $A \in \bigcup_{m<n} \mathcal{A}_m$.

Choose $\mathcal{A}_n$ to be a minimal (finite) subcover of $\{N_p : p \in K\}$.

Since $\max_{A\in\mathcal{A}_n}$ diam$(A) \le 2^{-n}$, $\mathcal{A}$ will be a base.

Since also each $\mathcal{A}_n$ is finite, Nt$(\mathcal{A}) = \aleph_0$.

Since diam$(A) <$ diam(B) for all $m > n$, $A \in \mathcal{A}_m$, and $B \in \mathcal{A}_n \setminus [K]^1$,

$$\text{if } \mathcal{A}_i \ni U \supsetneqq V \in \mathcal{A}_j, \text{ then } i \le j.$$

Since also each $\mathcal{A}_n$ is a minimal cover,

$$\text{if } \mathcal{A}_i \ni U \supsetneqq V \in \mathcal{A}_j, \text{ then } i < j.$$

Since also $\mathcal{A}_i \ni U \supsetneqq V \in \mathcal{A}_j$ and $i < j$ imply $U \supset \overline{V}$,

$$U \supsetneqq V \Rightarrow U \supset \overline{V} \text{ for all } U, V \in \mathcal{A}.$$

Finally, given a finite $\mathcal{F} \subset \mathcal{A}$ and a non-repeating sequence $(U_n)_{n<\omega}$ of elements of $\mathcal{A}$, it suffices to find some U_n with a strict superset in $\mathcal{A} \setminus \mathcal{F}$.

Since $(U_n)_{n<\omega}$ is non-repeating and each $\mathcal{A}_n$ is finite, we may pass to a subsequence $(V_n)_{n<\omega}$ of $(U_n)_{n<\omega}$ that diam$(V_n) \to 0$.

We may then pass to a subsequence $(W_n)_{n<\omega}$ such that $(\overline{W_n})_{n<\omega}$ converges to a singleton $\{p\}$ (in the (compact) Vietoris hyperspace).

Since $(W_n)_{n<\omega}$ is non-repeating, p is not an isolated point.

Hence, p has a neighborhoods $Y, Z \in \mathcal{A} \setminus \mathcal{F}$ such that $Y \subsetneq Z$.

For m sufficiently large, $W_m \subset Y \subsetneq Z$.

Let X be a compact space of uncountable weight κ. Without loss of generality, X is a subspace of $[0,1]^\kappa$.

Let $\mathcal{A}$ be a base of X of size κ and consisting only of nonempty open F_σ sets.

(To find such a base, take any base $\mathcal{Z}$ of size κ and, for each finite subcover of $\mathcal{Z}$, choose a refining finite cover by open F_σ sets; take $\mathcal{A}$ to be the union these refinements.)

Given a function f and a set I, let $f \upharpoonright I$ denote the restriction of f to $\mathrm{dom}(f) \cap I$. Given a set E of functions, let $E \upharpoonright I$ denote $\{f \upharpoonright I : f \in E\}$. Given a set $\mathcal{J}$ of sets of functions, let $\mathcal{J} \upharpoonright I = \{E \upharpoonright I : E \in \mathcal{J}\}$.

We say that $E \subset X$ is *supported* on a set I if, for all $p, q \in X$, if $p \upharpoonright I = q \upharpoonright I$, then $p \in E \Leftrightarrow q \in E$.

By compactness of X, every open F_σ set has a countable support.

Assume that there is a continuous surjection $h\colon 2^\lambda \to X$.

Let $(M_\alpha)_{\alpha<\kappa}$ be a long ω_1-approximation sequence with $\mathcal{A}, h \in M_0$.

Letting $\mathcal{A}_\alpha = \mathcal{A} \cap M_\alpha$, each $U \in \mathcal{A}_\alpha$ is supported on M_α.
Why? Each $U \in \mathcal{A} \cap M_\alpha$ is supported on some countable C. M_α knows this; hence, we may choose $C \in M_\alpha$; hence, $C \subset M_\alpha$.

For each $\alpha < \kappa$, $\mathcal{A}_\alpha \restriction M_\alpha$ is a base of $X \restriction M_\alpha$.

Why? Given $p \in X$, if R is an open product of rational intervals such that $p \in R$ and $R \cap X$ is supported on M_α, then $R \cap X$ is supported on a finite $F \subset M_\alpha$ and there is a closed product Q of rational intervals such that $p \in Q \subset R$ and $Q \cap X$ is suppported on F. M_α knows about a finite cover of $Q \cap X$ by elements of $\mathcal{A}$ with union contained in $R \cap X$. Hence, $p \in A \subset R \cap X$ and $A \in \mathcal{A} \cap M_\alpha$ for some A in this cover. Hence, $p \restriction M_\alpha \in A \restriction M_\alpha \subset (R \cap X) \restriction M_\alpha$ and $A \restriction M_\alpha$ is open in $X \restriction M_\alpha$ because A is supported on M_α.

We may choose $\mathcal{Y}_\alpha \subset \mathcal{A}_\alpha \restriction M_\alpha$ to be an efficient base of $X \restriction M_\alpha$. (Why? Every compact space with countable weight is metrizable.)

Because each $A \in \mathcal{A}_\alpha$ is supported on M_α, there is a unique $\mathcal{W}_\alpha \subset \mathcal{A}_\alpha$ such that $\mathcal{Y}_\alpha = \mathcal{W}_\alpha \restriction M_\alpha$.

Given E a subset of a poset P, let $\uparrow E = \bigcup\{p\uparrow : p \in E\}$.

Let $\mathcal{V}_\alpha = \mathcal{W}_\alpha \setminus \uparrow \mathcal{W}_{<\alpha}$ where $\mathcal{W}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{W}_\beta$.

Let $\mathcal{U}_\alpha = \{U \in \mathcal{V}_\alpha : \exists V \in \mathcal{V}_\alpha \ \ \overline{U} \subset V\}$.

Assume that $\min_{p \in X} \pi\chi(p, X) = \kappa$.

We claim that $\mathcal{U} = \mathcal{U}_{<\kappa}$ is a base of X with Noetherian type $\aleph_0$.

First, we show that $\mathcal{U}$ is a base.

Given $p \in A \in \mathcal{A}$, we need to find $U \in \mathcal{U}$ such that $p \in U \subset A$. Choose $\alpha < \kappa$ such that $A \in M_\alpha$. Then A is supported on M_α just as each $U \in \mathcal{U}_\alpha$ is, so it suffices to show that $\mathcal{U}_\alpha \restriction M_\alpha$ is a base of $X \restriction M_\alpha$.

$\mathcal{U}_\alpha$ is a downward-closed subset of $\mathcal{W}_\alpha$. Therefore, $\mathcal{U}_\alpha \restriction M_\alpha$ is a downard-closed subset of the base $\mathcal{W}_\alpha \restriction M_\alpha$. Hence, it suffices to show that $\mathcal{U}_\alpha \restriction M_\alpha$ covers $X \restriction M_\alpha$.

Because $\mathcal{A}_{<\alpha}$ is too small to contain a local π-base, M_α knows about a finite cover of X by elements of $\mathcal{A} \setminus \uparrow \mathcal{A}_{<\alpha}$. We have $p \in T \in M_\alpha$ for some T in this cover.

$T \restriction M_\alpha$ is open, so we may choose $R, S \in \mathcal{W}_\alpha \restriction M_\alpha$ such that

$$p \restriction M_\alpha \in \overline{R} \subset S \subset T \restriction M_\alpha.$$

R meets all the requireements for being in $\mathcal{U}_\alpha \restriction M_\alpha$.

It remains to show that $\mathrm{Nt}(\mathcal{U}) = \aleph_0$.

For this, we must actually use the continuous surjection $h\colon 2^\lambda \to X$.

Let $\mathcal{B}$ denote the clopen algebra $\mathrm{Clop}(2^\lambda)$.

Since $\mathcal{W}_\alpha \restriction M_\alpha$ is an efficient base, for each α and $W \in \mathcal{W}_\alpha$, there is an $E_{\alpha,W} \in \mathcal{B} \cap M_\alpha$ such that

$$h^{-1}[\overline{W}] \subset E_{\alpha,W} \subset \bigcap\{h^{-1}[Z] : \overline{W} \subset Z \in \mathcal{W}_\alpha\}$$

because only there are only finitely many Z as above.

Letting $\mathcal{E}_\alpha = \{E_{\alpha,W} : W \in \mathcal{W}_\alpha\}$, we have $\mathrm{Nt}(\mathcal{E}_\alpha) = \aleph_0$.

Why? If $E_{\alpha,R} \subsetneq E_{\alpha,S_m} \neq E_{\alpha,S_n}$ for all $m < n < \omega$, then, for all $m < \omega$ and $\overline{S_m} \subset T \in \mathcal{W}_\alpha$, we have $R \subset T$. By the definition of efficient base, there are infinitely many T as above, in contradiction with $\mathrm{Nt}(\mathcal{W}_\alpha) = \aleph_0$.

Let $\mathcal{D}_\alpha = \{E_{\alpha,U} : U \in \mathcal{U}_\alpha\}$. We have $\mathrm{Nt}(\mathcal{D}_\alpha) = \aleph_0$ because $\mathcal{D}_\alpha \subset \mathcal{E}_\alpha$.

Let $\mathcal{C} = \mathcal{B} \cap \uparrow \{h^{-1}[U] : U \in \mathcal{U}\}$.

Let $\mathcal{C}_\alpha = \mathcal{C} \cap M_\alpha$. Note that $\mathcal{D}_\alpha \subset \mathcal{C}_\alpha$.

Letting $\mathcal{D} = \mathcal{D}_{<\kappa}$, we claim that $\mathrm{Nt}(\mathcal{D}) = \aleph_0$.

To prove this, it suffices to show that, for all $\alpha < \kappa$ and $H \in \mathcal{C}_{<\alpha}$,

1. $\mathcal{C}_\alpha \subset \uparrow \mathcal{D}_\alpha$,

2. $H\uparrow \cap \mathcal{D}_{<\alpha}$ is finite, and

3. $H\uparrow \cap \mathcal{D}_\alpha = \varnothing$.

To be continued...