Symmetries of Homogeneous Structures

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Homogeneous structures

Definition

A structure $\mathcal{S}$ is (ultra)homogeneous *if every isomorphism between finite substructures extends to an automorphism of the entire structure.*

Example

Fraïssé classes		**Fraïssé limits**
Finite linear orders	$\to$	*Rationals* $\mathbb{Q}$
Finite graphs	$\to$	*Rado graph* $\mathcal{R}$
Finite graphs omitting K_n	$\to$	K_n*-free graph*
Finite metric spaces with rational dist.	$\to$	*Rational Urysohn space* $\mathcal{U}$
And many more....		

Symmetries of Homogeneous Structures

Definition

Let $\mathcal{S} = (E, ..)$ *be an homogeneous structure and consider*

$$Aut(\mathcal{S}) = \text{ automorphisms of } \mathcal{S}$$

By "Symmetries", we mean the overgroups $\mathcal{G}$ *of* $Aut(\mathcal{S})$*:*

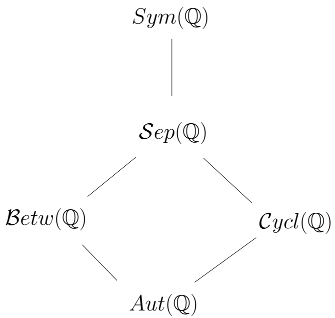
$$Aut(\mathcal{S}) \leq \mathcal{G} \leq Sym(\mathcal{S})$$

Reducts of the Rationals $\mathbb{Q} = (\mathbb{Q}, <)$

P. Cameron (76)

The closed subgroups of $Sym(\mathbb{Q})$ containing $Aut(\mathbb{Q})$ (the reducts) are:

- $Aut(\mathbb{Q})$
- $\mathcal{B}etw(\mathbb{Q})$, the group of automorphisms and anti-automorphisms.
- $\mathcal{C}ycl(\mathbb{Q})$, the group of cycling automorphisms.
- $\mathcal{S}ep(\mathbb{Q})$ generated by the previous two groups.
- $Sym(\mathbb{Q})$



Preserving Copies

Exercise

If $f : \mathbb{Q} \to \mathbb{Q} \in Sym(\mathbb{Q})$ *preserves copies, then what can* f *be?*

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Definition

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- x *is of type OP if* $(\forall y)[x < y \implies f(x) < f(y)] \wedge [y < x \implies f(y) < f(x)]$.
- x *is of type ROP if* $(\forall y)[x < y \implies f(y) < f(x)] \wedge [y < x \implies f(y) > f(x)]$.

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If $f : \mathbb{Q} \to \mathbb{Q} \in Sym(\mathbb{Q})$ preserves copies, then:

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Corollary

If $f : \mathbb{Q} \to \mathbb{Q} \in Sym(\mathbb{Q})$ preserves copies, then:

$$(\forall x)\ [x \text{ is } OP] \text{ or } (\forall x)[x \text{ is } ROP]$$

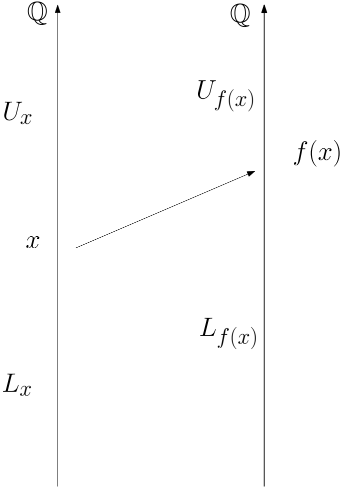
Thus f is order preserving or reverse order preserving.

Preserving Copies

Proof.

Case 1: $U_x \cap f^{-1}(U_{f(x)})$ is scattered.

□



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Case 1: $U_x \cap f^{-1}(U_{f(x)})$ is scattered.
Claim: $U_x \cap f^{-1}(U_{f(x)})$ is empty!

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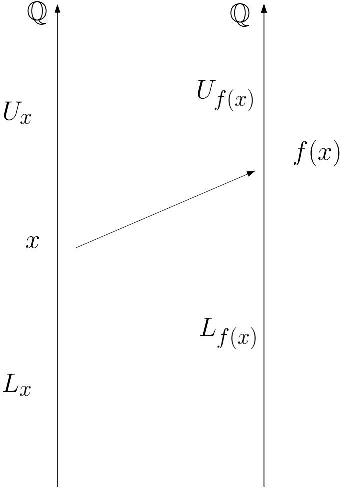

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Claim: $U_x \cap f^{-1}(U_{f(x)})$ is empty!
Else if there is such an x', consider two copies:

$$C_0 \subseteq U_x \cap f^{-1}(L_{f(x)} \cap (x, x'))$$
$$C_1 \subseteq U_x \cap f^{-1}(L_{f(x)} \cap [x', \infty))$$

Then $C_0 \cup \{x'\} \cup C_1$ is a copy, but the image by f has a largest element, a contradiction.
Similarly $L_x \cap f^{-1}(L_{f(x)})$ is empty, and thus x is *ROP*.

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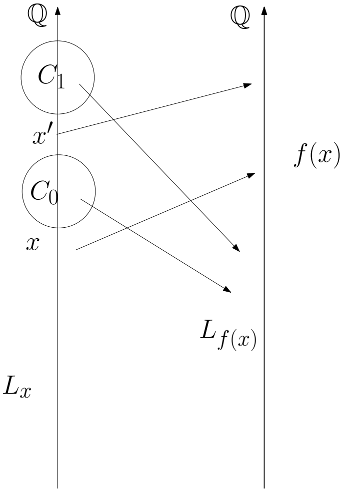

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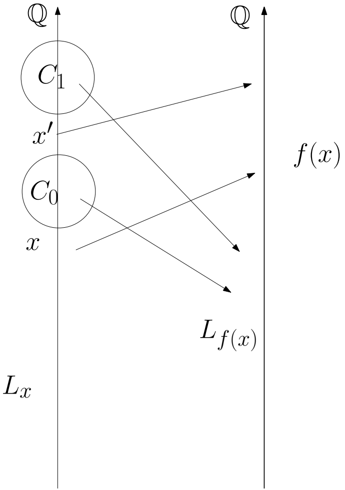
Similarly $L_x \cap f^{-1}(L_{f(x)})$ is empty, and thus x is *ROP*.
Case 2: $U_x \cap f^{-1}(U_{f(x)})$ is NOT scattered.
In this case one shows x is *OP*. □



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What is $Aut(\Gamma_{\mathcal{S}})$*?*

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Remark

$Aut(\Gamma_{\mathbb{Q}}) = \mathcal{B}etw(\mathbb{Q})$.

K_n-free graph $\mathcal{H}_n = (V, E)$

Theorem (Thomas (91))

There is no closed groups between $Aut(\mathcal{H}_n)$ and $Sym(\mathcal{H}_n)$

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Theorem

$Aut(\Gamma_{\mathcal{H}_n}) = Aut(\mathcal{H}_n)$

Proof of $Aut(\Gamma_{\mathcal{H}_n}) = Aut(\mathcal{H}_n)$.

(Triangle-Free $\mathcal{H}_3 = (V, E)$) Let $f : \mathcal{H}_3 \to \mathcal{H}_3$ preserve copies (and conversely), and suppose wlog some edge (a, b) is mapped to a non edge.

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- Define a new graph $\mathcal{H}'_3 = (V, E')$ by:

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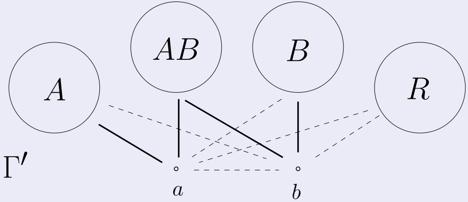

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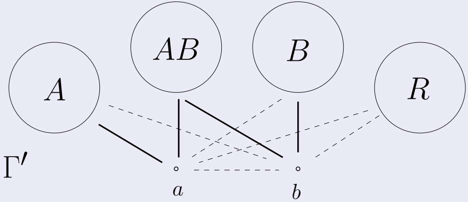


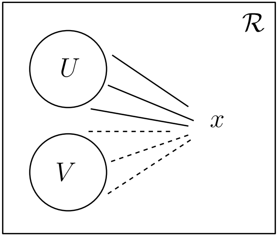
- So the same is true in $\mathcal{H}_3$. But in $\mathcal{H}_3$, 2&3 $\implies$ 1, a contradiction. □

Rado Graph

Folklore

The Rado graph $\mathcal{R}$ is the (unique) countable graph with the property that:

For all finite disjoint $U, V \subseteq \mathcal{R}$, there is a vertex x connected to all vertices of U and none of V.

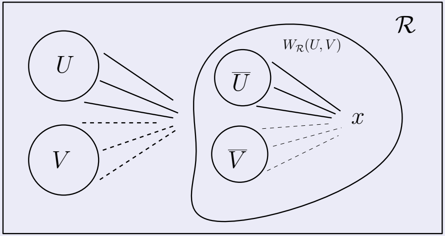


Definition

Let $W_{\mathcal{R}}(U, V)$ be the collection of all these witness x

Basic Properties

- $W_{\mathcal{R}}(U, V)$ *is a copy of* $\mathcal{R}$.
  Proof: $W_{\mathcal{R}}(U \cup \overline{U}, V \cup \overline{V}) = W_{\mathcal{R}}(U, V) \cap W_{\mathcal{R}}(\overline{U}, \overline{V})$.



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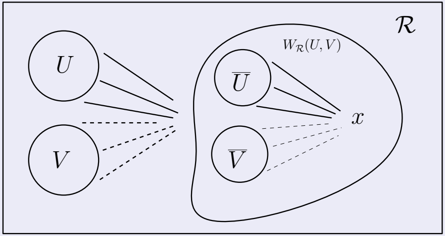

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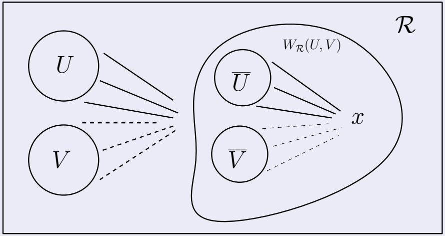

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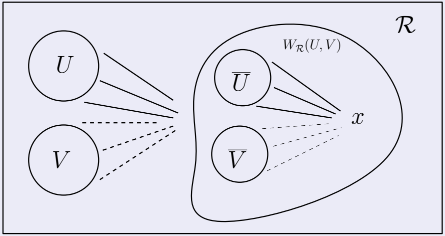

- *$\mathcal{R}$ is universal: it embeds all finite (and countable) graphs.*
- *$\mathcal{R}$ is unique (up to isomorphism).*
- *$\mathcal{R}$ exists: Fraïssé limit of all finite graphs.*

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$\mathcal{R}$ is (strongly) indivisible:
If $\mathcal{R} = A \cup B$, then one of A or B IS the Rado graph.

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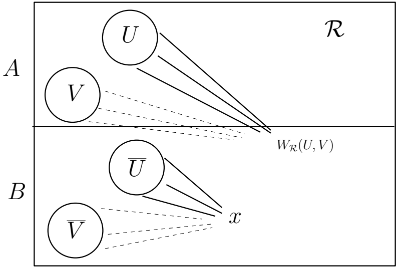
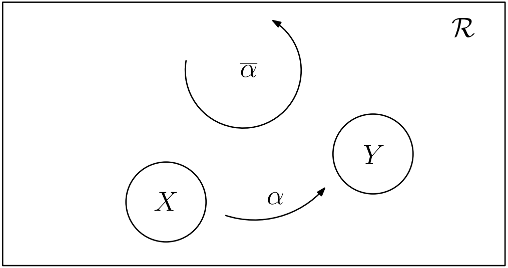
Proof.

If A is not Rado with bad pair U, V, then $W_{\mathcal{R}}(U, V) \subseteq B$. But $W_{\mathcal{R}}(U \cup \overline{U}, V \cup \overline{V}) = W_{\mathcal{R}}(U, V) \cap W_{\mathcal{R}}(\overline{U}, \overline{V})$. □



Automorphism Group $Aut(\mathcal{R})$

- $\mathcal{R}$ is homogeneous:
  Any finite partial automorphism $\alpha : X \to Y$ extends to a full automorphism $\overline{\alpha}$.



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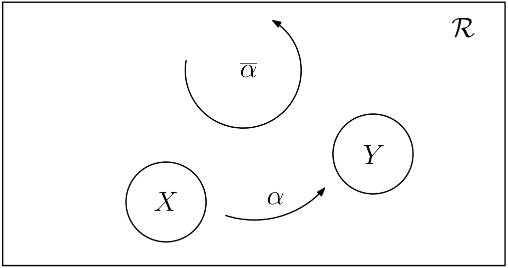

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Switching

For $X \subset \mathcal{R}$, consider the new graph $S(X)$ on the same vertex set as $\mathcal{R}$, but adjacencies between X and X^c are switched.
$\mathcal{S}(\mathcal{R})$ consists of all switching automorphisms, that is graph isomorphism $\alpha : \mathcal{R} \to S(X)$ for some X.

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Big Group

Call $\mathcal{B}(\mathcal{R})$, the big group, generated by $\mathcal{D}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$.

S. Thomas (91)

The closed subgroups of $Sym(\mathcal{R})$ containing $Aut(\mathcal{R})$ (the reducts) are:

- $Aut(\mathcal{R})$
- $\mathcal{D}(\mathcal{R})$, the group of automorphisms and anti-automorphisms.
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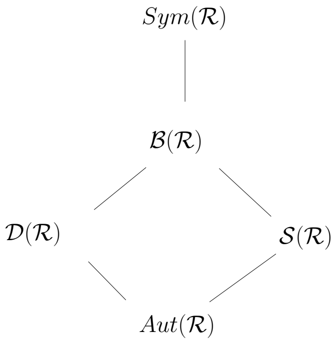
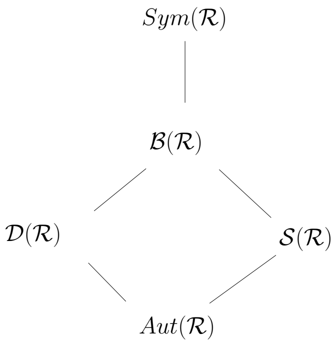
- $\mathcal{B}(\mathcal{R})$, the group of switching automorphisms and anti-automorphisms.
- $Sym(\mathcal{R})$



Observation

- $Aut(\mathcal{R})$ *is 1-transitive, not 2-transitive.*
- $\mathcal{S}(\mathcal{R})$ *is 2-transitive, not 3-transitive.*
- $\mathcal{D}(\mathcal{R})$ *is 2-transitive, not 3-transitive.*
- $\mathcal{B}(\mathcal{R})$ *is 3-transitive, not 4-transitive.*
- $Sym(\mathcal{R})$ *is highly transitive.*



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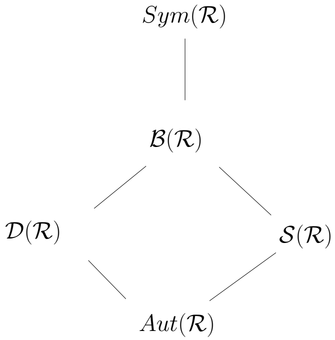

Cameron

Any overgroup of $Aut(\mathcal{R})$ not contained in $\mathcal{B}(\mathcal{R})$ is highly transitive.

Question

What about $Aut(\Gamma_{\mathcal{R}})$*?*

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Theorem

Any bijection $f : X \to X'$ between two scattered sets X and X' extends to an automorphism of $\Gamma_{\mathcal{R}}$.

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Theorem

Any bijection $f : X \to X'$ *between two scattered sets* X *and* X' *extends to an automorphism of* $\Gamma_{\mathcal{R}}$*.*

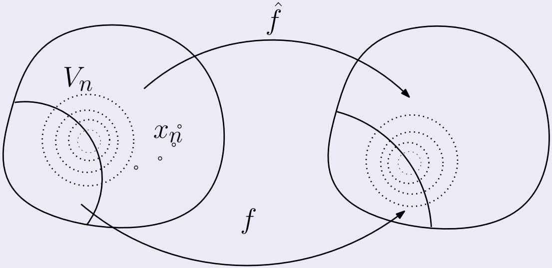
Corollary

$Aut(\Gamma_{\mathcal{R}})$ *is highly transitive, and thus cannot be any of the reducts.*

Proof.

Let $f : X \to X'$ be a bijection between scattered sets.
Write $V = \bigcup_n V_n$, and list $V \setminus X =< x_n : n \in \omega >$.
Extend f to $\hat{f} = \bigcup_n f_n$ such that for each n:

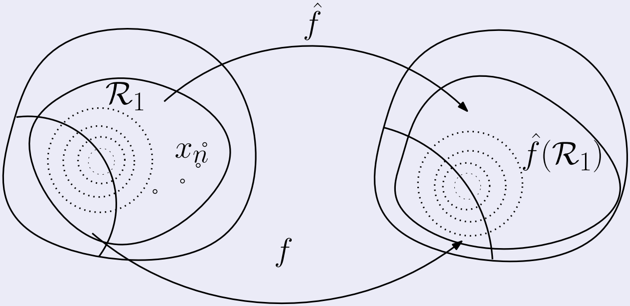
1. $dom(f_n) = C_n \supseteq V_n$
2. There is $k(n)$ so that for all $k \geq k(n)$ the type of x_k over V_n is the same as $\hat{f}(x_k)$ over $\hat{f}(V_n)$.



Proof.

(Cont'd) To show that it works, let $\mathcal{R}_1$ be a copy of $\mathcal{R}$.
We show that $\hat{f}(\mathcal{R}_1)$ is also a copy.

We need to realize every type in $\hat{f}(\mathcal{R}_1)$.



□

Finite Variations

Definition

- $Aut(\Gamma_{\mathcal{R}}) =$

$$\{\sigma \in Sym(\mathcal{R}) : \forall E \in \Gamma_{\mathcal{R}}\ E\sigma \ and\ E\sigma^{-1} \in \Gamma_{\mathcal{R}}\}$$

- $FAut(\Gamma_{\mathcal{R}}) =$

$$\{\sigma \in Sym(\mathcal{R}) : \exists F\ finite\ \ \forall E \in \Gamma_{\mathcal{R}}\ (E \setminus F)\sigma\ and\ (E \setminus F)\sigma^{-1} \in \Gamma_{\mathcal{R}}\}$$

- $Aut^{*}(\Gamma_{\mathcal{R}}) =$

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- $Aut^*(\Gamma_{\mathcal{R}}) =$

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Proposition

$\mathcal{S}(\mathcal{R}) \not\leq FAut(\Gamma_{\mathcal{R}})$, but $\mathcal{S}(\mathcal{R}) \leq Aut^*(\Gamma_{\mathcal{R}})$

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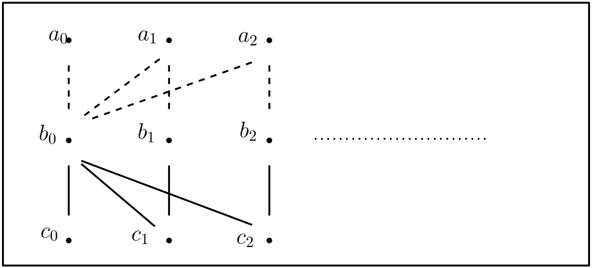

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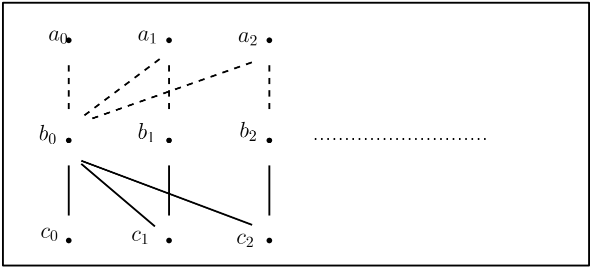

1. $\forall n \, \forall k \leq n \; a_n \not\sim b_k$ and $c_n \sim b_k$

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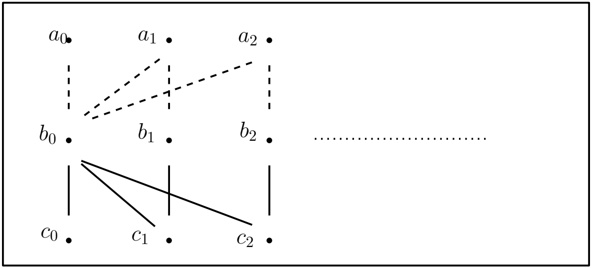

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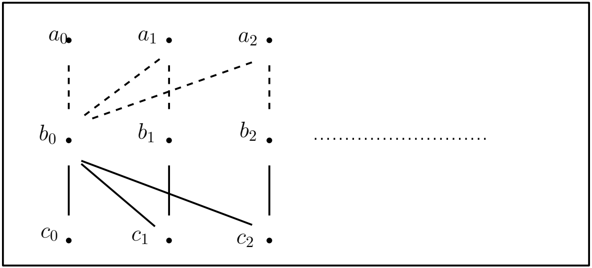

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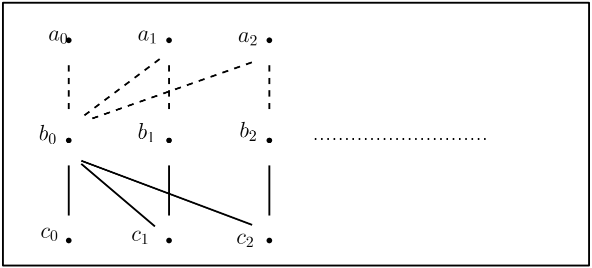

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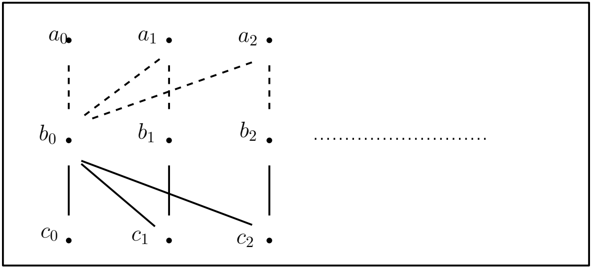

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3. $S(C)$ is the Rado graph.

- In $S(C)$, b_n is isolated in E_n.
- For any finite set F, choose n large enough so that $E_n = E_n \setminus F$.

Then E_n is a copy in $\mathcal{R}$, but E_n is not a copy in $S(C)$.

$\mathcal{S}(\mathcal{R}) \leq Aut^*(\Gamma_{\mathcal{R}})$

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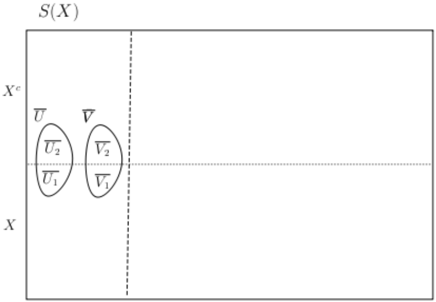

If $S(E) \notin \mathcal{H}$, then $W_E(\overline{U_2} \cup \overline{V_1}, \overline{U_1} \cup \overline{V_2}) \subseteq X$

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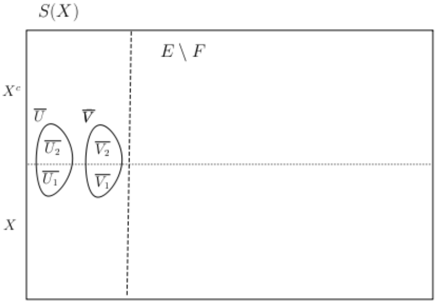

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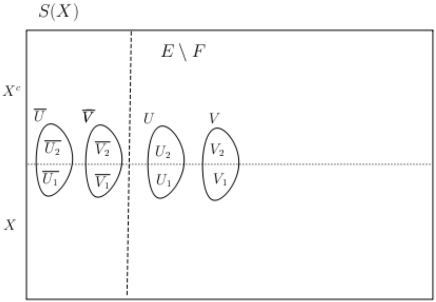

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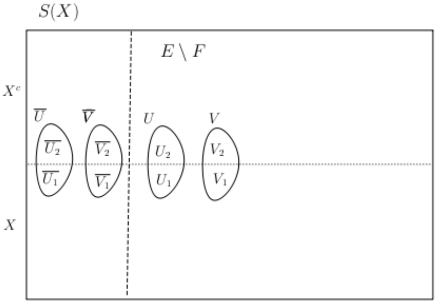

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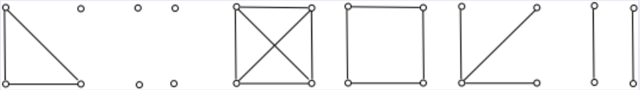
So $W_{S(X)}(U, V) \cap (E \setminus F) \neq \emptyset$

Hypergraph of copies

$\mathcal{B}(\mathcal{R}) < Aut^*(\Gamma_{\mathcal{R}})$

Proof

The orbit of K_4 under the action of $\mathcal{B}(\mathcal{R})$ is



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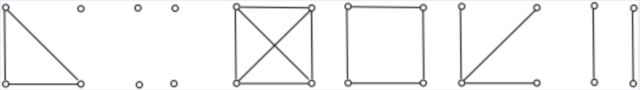
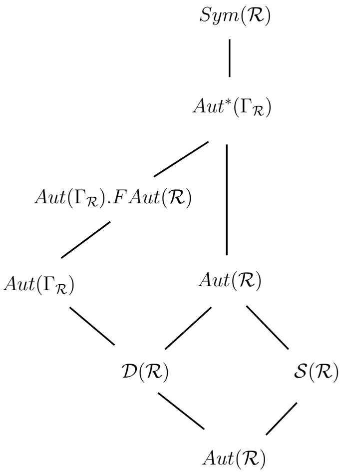


But the orbit under $Aut^*(\Gamma_{\mathcal{R}})$ contains all graphs on 4 elements.

Some Overgroups of $Aut(\mathcal{R})$



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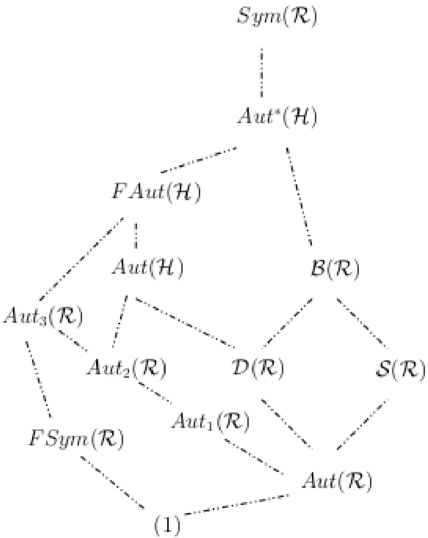
c) $Aut_3(\mathcal{R})$, the group of permutations which change only a finite number of adjacencies at all but finitely many vertices;


$Sym(\mathcal{R})$
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$FAut(\mathcal{H})$
$Aut(\mathcal{H})$
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$Aut_2(\mathcal{R})$
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(1)

Question

- *What about the automorphism group of the neighbouring filter* $\mathcal{F}(\mathcal{R})$*?*

  [$\mathcal{F}(\mathcal{R})$ $=$ *the filter generated by the (open or closed) neighbourhoods in* $\mathcal{R}$ *]*

Question

- *What about the automorphism group of the neighbouring filter* $\mathcal{F}(\mathcal{R})$*?*

  [$\mathcal{F}(\mathcal{R})$ = *the filter generated by the (open or closed) neighbourhoods in* $\mathcal{R}$]

- *What about the rational Urysohn space, random partial order, or other homogeneous structures?*