How many siblings do you have?

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Definition (Siblings)

- Write

$$\mathcal{A} \leq \mathcal{B} \text{ if there is an embedding from } \mathcal{A} \text{ to } \mathcal{B},$$
$$\mathcal{A} \equiv \mathcal{B} \text{ if both } \mathcal{A} \leq \mathcal{B} \text{ and } \mathcal{B} \leq \mathcal{A}.$$

  In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic*, or *siblings*, or that $\mathcal{B}$ is a *sibling* of $\mathcal{A}$ (and vice-versa).
- *sib*($\mathcal{A}$) denotes the number of siblings, up to isomorphism.

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Example

- (Cantor-Bernstein-Schroeder) $sib(X) = 1$ for any set X.
- (Vectors Spaces) $sib(\mathcal{V}) = 1$ for any vector space $\mathcal{V}$.

Example (Linear Order)

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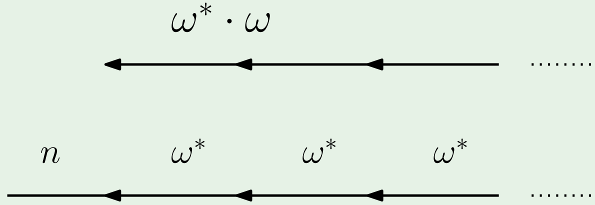

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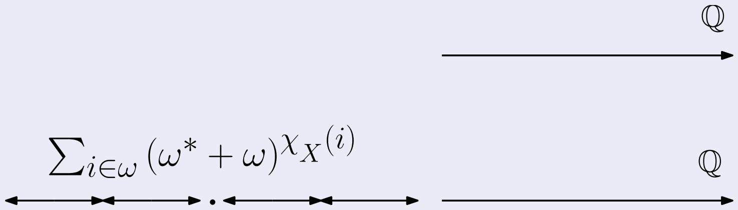

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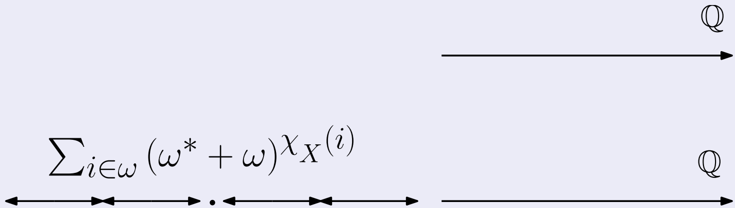

Corollary (Non Scattered)

$sib(C) = 2^{\aleph_0}$ *for all non-scattered countable chains.*

Conjecture (Thomassé)

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Theorem (Linear Orders)

We verify this conjecture for any countable chain C.

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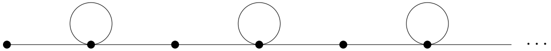

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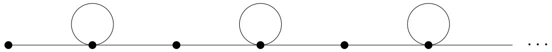
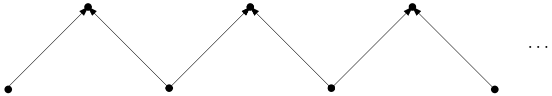


This is also the case for connected posets, as we may simply consider a one way infinite fence, which has two equimorphic siblings:



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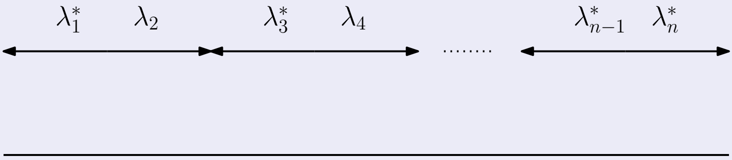

$\square$

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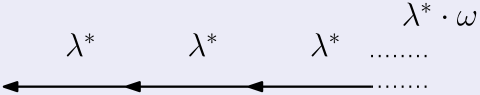

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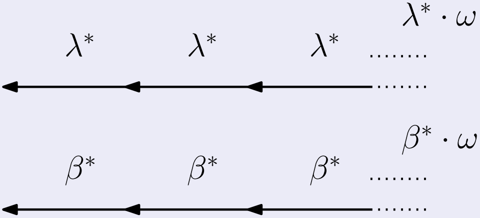

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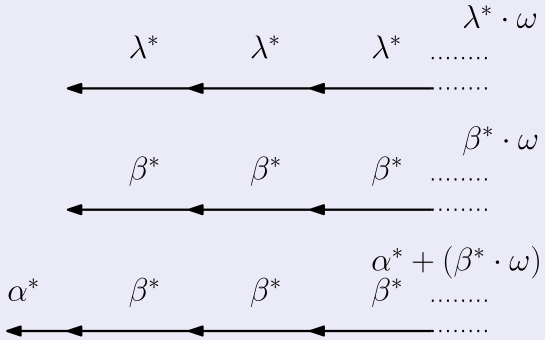

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If a chain is of the form $C = \sum_{i \in \omega} \kappa_i^*$ *(or its reverse) where the* κ_i*'s form a strictly increasing chain of cardinals (or even ordinals of strictly increasing cardinalities), then* $sib(C) \geq max\{2^{\aleph_0}, sup_i\{\kappa_i\}\}$.

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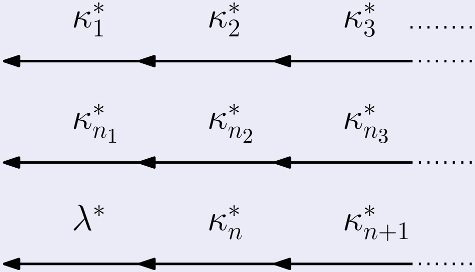

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Proof.

Proceed by induction on Hausdorff rank. For $x, y \in C$, the equivalence relations:

- $x \equiv_0 y$ if the interval $[x, y]$ is finite.
- $x \equiv_{\alpha+1} y$ if the interval $[x/\equiv_\alpha, y/\equiv_\alpha]$ is finite in $C/\equiv_\alpha$.
- $\equiv_\beta := \bigcup_{\alpha<\beta} \equiv_\alpha$.

Then the Hausdorff rank of C, written $h(C)$, is the least ordinal α such that $\equiv_\alpha = \equiv_{\alpha+1}$. □

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- *A surordinal is pure if and only if it can be written as a sum $\sum_{n<\omega^*} C_n$ where each C_n has order type ω^{α_n} and the sequence $(\alpha_n)_{n<\omega}$ is non-decreasing.*
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Proposition

Neither $(\omega^* + \omega) \cdot \omega$ nor $(\omega^* + \omega) \cdot \omega^*$ are embeddable into a chain C if and only if C is a finite sum of surordinals and reverse of surordinals.

Proposition

Let C be a surordinal. Then:

1. *$sib(C) = 1$ if and only if either C is an ordinal, ω^*, or C is not pure but the sequence in a component is stationary, that is $C = \omega^\alpha \cdot \omega^* + \omega^\beta + \gamma$ with $\alpha + 1 \leq \beta$ and γ ordinal.*

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3. $sib(C) = |C'|^{\aleph_0}$ *if the sequence in a component C' of C is non-stationary.*

Theorem (Scattered Chains with Few Siblings)

Let C be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1. $sib(C) = \kappa$ *and* C *is scattered;*
2. $\kappa = 1$, *or* $\kappa \geq \aleph_0$ *and* C *is a finite sum of surordinals and of reverse of surordinals, and if* $C = \sum_{j<m} D_j$ *is such a sum with* m *minimum then* $max\{sib(D_j) : j < m\} = \kappa$.

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Corollary

When C is countable, then $sib(C) = 1$, $\aleph_0$, or $2^{\aleph_0}$.

Corollary (Scattered Chains with Few Siblings)

Let C be a chain. Then:

1. *C is scattered and sib$(C) = \kappa < 2^{\aleph_0}$ if and only if C is a finite sum $\sum_{i<n} C_i$ of ordinals, surordinals of the form $\omega^\alpha \cdot \omega^* + \omega^\beta$ with $\alpha + 1 \leq \beta$, surordinals of the form $\omega^\alpha \cdot \omega^*$ and reverse of such chains. Furthermore if the number of parts C_i of this sum such that C_i or its reverse is of the form $\omega^\alpha \cdot \omega^*$ with $\alpha \geq 1$ is minimum, then κ is the maximum cardinality of these parts.*
2. *sib(C) is finite and C is scattered if and only if C is a finite sum of ordinals, surordinals of the form $\omega^\alpha \cdot \omega^* + \omega^\beta$ with $\alpha + 1 \leq \beta$, and their reverse. In which case, sib$(C) = 1$.*

Theorem (Chains with Few Siblings)

Let C be any chain and $\kappa < 2^{\aleph_0}$. Then the following are equivalent:

1. $sib(C) = \kappa$.
2. $C = \sum_{i \in D} C_i$, *where:*
   - D *is dense (singleton or infinite),*
   - *each* C_i *is scattered,*
   - $sib(C_i) = 1$ *for all but finitely many* $i \in D$,
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Proposition

If $C = \sum_{i \in D} C_i$ where each C_i is scattered and D is a countably infinite dense chain, then $sib(C) \geq 2^{\aleph_0}$.

Example (Dushnik and Miller (40))

It is possible to have $C = \sum_{i \in \mathbb{R}} C_i$, and $sib(C) = 1$.

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Solution

$\mathbb{R}$ can be decomposed into two disjoint dense subsets E and F such that

$$g(E) \cap F \neq \emptyset \text{ and } g(F) \cap E \neq \emptyset$$

for any non-identity order preserving map $g : \mathbb{R} \to \mathbb{R}$.

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Thus if $C = \sum_{i \in \mathbb{R}} C_i$, where:

$$\begin{cases} |C_i| = 2 \text{ if } i \in E \\ |C_i| = 1 \text{ if } i \notin E \ (i \in F), \end{cases}$$

then C itself is embedding rigid.

Problem

Suppose that $C = \sum_{i \in D} C_i$*, where:*

- D *is embedding rigid,*
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- $sib(C_i) = 1$ *for all but finitely many* $i \in D$*, and*
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Does it follow that $sib(C) = \kappa$*?*

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Suppose that $C = \sum_{i \in D} C_i$*, where* D *and every* C_i *are embedding rigid, is* C *necessarily embedding rigid?*

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Question

Are there κ-dense embedding rigid chains of size κ for each regular and uncountable cardinal κ?