Radality and compactifications

Robert Leek
DPhil candidate, University of Oxford
robert.leek@maths.ox.ac.uk
www.maths.ox.ac.uk/people/profiles/robert.leek

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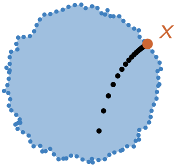
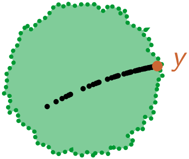
Part I

Internal characterisation of radiality



Fréchet-Urysohn space

$(x_n)_{n<\omega} \to x$



Radial space

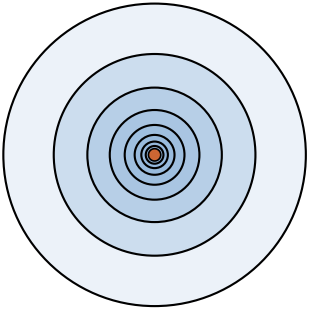
$(y_\alpha)_{\alpha<\gamma} \to y$

Definition

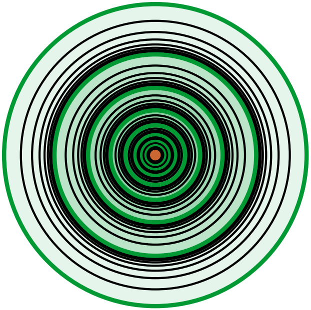
X is Fréchet-Urysohn at x if whenever $x \in \overline{A}$, there exists a sequence $(x_n)_{n<\omega}$ in A that converges to x.

Definition

Y is radial at a point y if whenever $y \in \overline{B}$, there exists a *transfinite* sequence $(y_\alpha)_{\alpha<\gamma}$ in B, for some ordinal γ, that converges to y.



First countable space



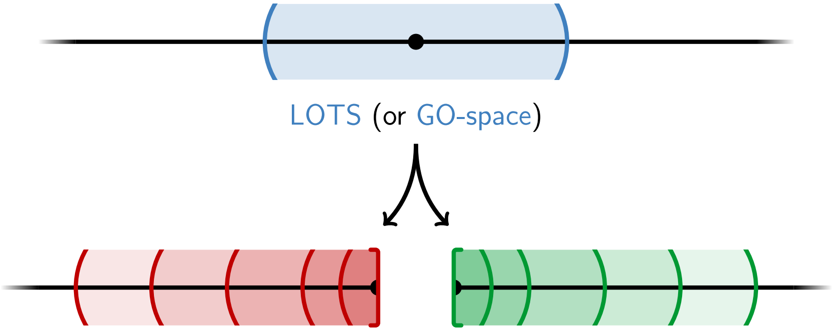
Well-based space

Definition

A point in X is said to be well-based if it has a well-ordered neighbourhood base with respect to $\supseteq$.

Definition

LOTS := Linearly-Ordered Topological Space
GO-space := Generalised-Ordered space = Subspaces of LOTS



Definition

A spoke for a point is a well-based subspace containing that point.

Definition

A collection of spokes $\mathscr{S} = (S_i)_{i \in I}$ for a point x is a *spoke system* for x if

$$\mathscr{B} := \left\{ \bigcup_{i \in I} B_i : \forall i \in I, B_i \in \mathscr{N}_x^{S_i} \right\}$$

is a neighbourhood base for x.

Theorem

Every point with a spoke system is radial.

Proof.

Let $(S_i)_{i \in I}$ be a spoke system for x and let $x \in \overline{A}$. If for all $i \in I, x \notin \overline{A \cap S_i}^{S_i}$, then pick $U_i \in \mathscr{N}_x^{S_i}$ such that $A \cap U_i = \emptyset$. Then $U := \bigcup_{i \in I} U_i$ is a neighbourhood of x missing A, which is a contradiction. Thus $x \in \overline{A \cap S_i}^{S_i}$ for some $i \in I$, so since S_i is well-based at x, we can find a convergent transfinite sequence inside $A \cap S_i$. □

Definition

A transfinite sequence $(x_\alpha)_{\alpha<\lambda}$ converges strictly to a point x if it converges to that point and x is not in the closure of any initial segment; that is, $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$, for all $\beta < \lambda$.

Lemma

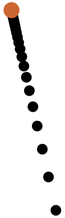
If X is radial at x and $x \in \overline{A}$, then there exists an injective, transfinite sequence in A that converges strictly to x.

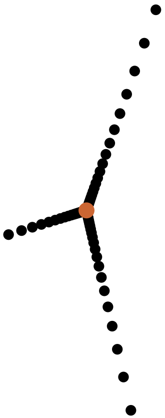
Lemma

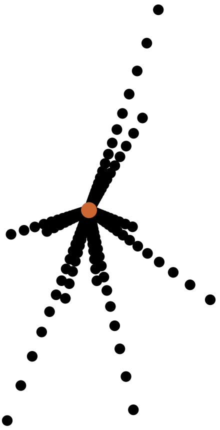
Let $(x_\alpha)_{\alpha<\gamma}$ be an injective transfinite sequence that converges strictly to x. Then $S := \{x\} \cup \{x_\alpha : \alpha < \gamma\}$ is a spoke for x.

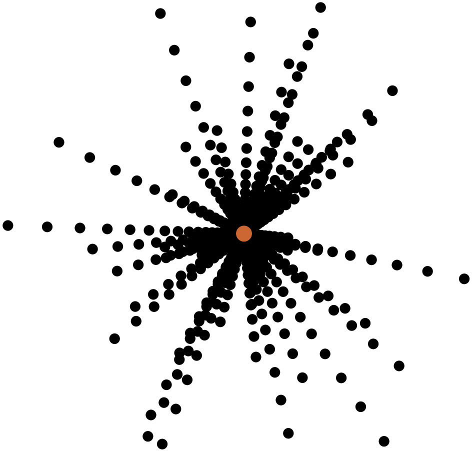
Proof.

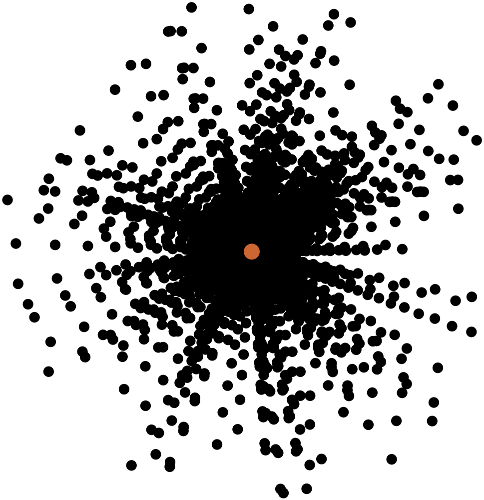
$\{\{x\} \cup \{x_\alpha : \alpha \in [\beta, \gamma)\} : \beta < \gamma\}$ is a neighbourhood base for x with respect to S. □

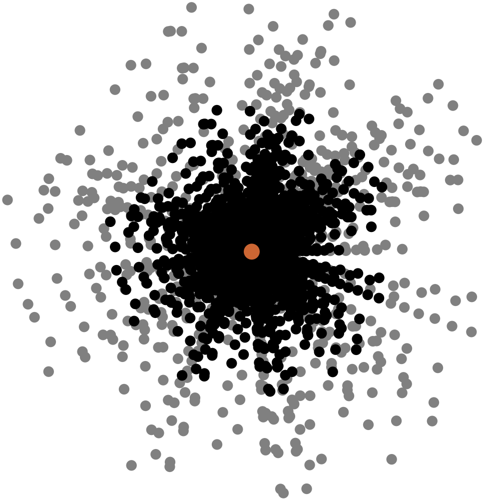


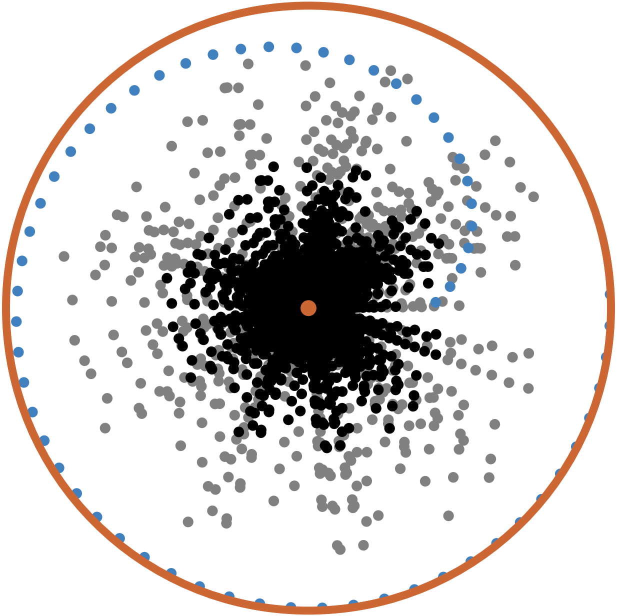












Theorem

For a point x in a topological space X, the following are equivalent:

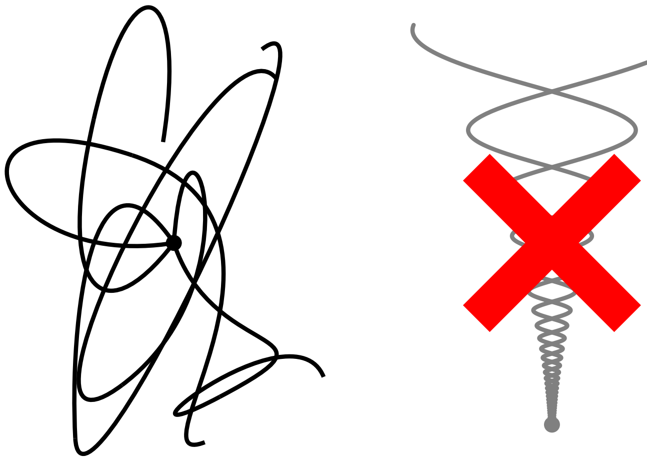
1. *X is radial at x.*
2. *X has a spoke system $(S_i)_{i\in I}$ at x such that for distinct $i,j\in I, x\notin\overline{(S_i\cap S_j)\setminus\{x\}}$.*

Proof.

If X is radial at x and not isolated, define

$$\mathscr{T} := \{f:\lambda\to X\setminus\{x\} : \lambda\leq |X|, f \text{ is injective and } f\to x \text{ strictly}\}$$
$$\mathscr{A} := \{\mathscr{F}\subseteq\mathscr{A} : \forall f,g\in\mathscr{F} \text{ distinct}, f^{-1}[\mathrm{ran}(g)] \text{ is bdd. in } \mathrm{dom}(f)\}$$

By Zorn's lemma, pick a maximal $\mathscr{F}\in\mathscr{A}$ and define for all $f\in\mathscr{F}, Y_f := \{x\}\cup\mathrm{ran}(f)$. Then by maximality, $(S_f)_{f\in\mathscr{F}}$ is a spoke system for x. Moreover, for all $f,g\in\mathscr{F}$ distinct, $x\notin\overline{(S_f\cap S_g)\setminus\{x\}}$ by strict convergence and since $\mathscr{F}\in\mathscr{A}$. □



For more information on this part, see:
http://arxiv.org/abs/1401.6519

Part II

Applications

Lemma

If X is a compact Hausdorff space and $x \in X$ is radial, then x has a closed spoke system $\mathscr{S} = (S_i)_{i \in I}$; i.e., S_i is closed in X for each $i \in I$.

Proof.

Let $\mathscr{S} = (S_f)_{f \in \mathscr{F}}$ be as in the proof of the existence of a spoke system for a radial point. Define $T_f := \{x\} \cup \bigcup_{\alpha \in \mathrm{dom}(f)} \overline{f[\alpha]}$ for each $f \in \mathscr{F}$. Then $\{T_f \setminus \overline{f[\alpha]} : \alpha \in \mathrm{dom}(f)\}$ is a neighbourhood base for x with respect to x. Also, T_f is compact and hence closed. □

From now on, assume X is a non-compact, locally compact Hausdorff space.

Definition

The one-point compactification of X is denoted by αX, with the point at infinity denoted by $\star$.

Lemma

If X is a space, $Y \subseteq X$ is open and radial and X is radial on $X \setminus Y$, then X is radial.

Lemma

Radiality is preserved under closed (or even pseudo-open) images.

Corollary

If X is radial, then X has a radial compactification if and only if αX is radial at $\star$.

Definition

The cardinal $\mathfrak{a}$ is the smallest size of an almost-disjoint family on ω. Note that $\aleph_1 \leq \mathfrak{a} \leq \mathfrak{c}$.

Proposition

Let $\mathscr{A}$ be an almost-disjoint family on ω and consider the Moore-Mrowka space $\Psi(\mathscr{A})$.

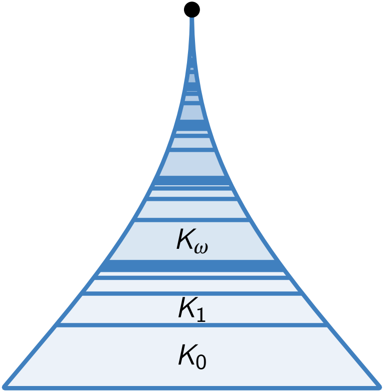
- $\Psi(\mathscr{A})$ *is first-countable.*
- *If $\mathscr{A}$ is maximal then $\alpha\Psi(\mathscr{A})$ is not radial at $\star$.*
- *If $|\mathscr{A}| < \mathfrak{a}$, (more generally, $\mathscr{A}$ is nowhere-mad), then $\alpha\Psi(\mathscr{A})$ is Fréchet-Urysohn.*

Definition

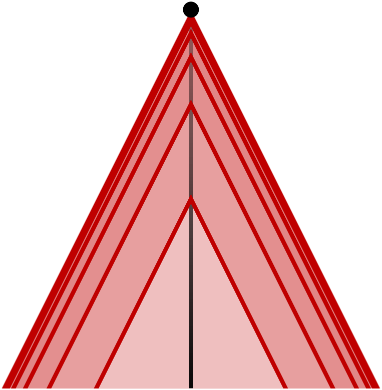
If $P \subseteq X$ is a closed, non-compact subspace which is an ascending union of compact subsets, we say that P is a *path to infinity*. We denote the collection of these by $\mathbf{P}^\infty(X)$.

Lemma

$P \subseteq X$ is a path to infinity if and only if $P \cup \{\star\}$ is a closed spoke of $\star$ in αX.



$P = \bigcup_{\alpha<\lambda} K_\alpha, K_\alpha \subseteq X$ is compact



f 1-1, $f \to \star$ strictly, $\mathrm{ran}(f) \subseteq X$
$P(f) := \bigcup_{\alpha \in \mathrm{dom}(f)} \overline{f[\alpha]}$

Theorem

The following are equivalent:

1. *αX is radial at $\star$.*
2. *For every $A \subseteq X$ with non-compact closure, an ascending union $(K_\alpha)_{\alpha<\lambda}$ of compact subsets of X such that $K_\alpha = \overline{A \cap K_\alpha}$ for all $\alpha < \lambda$ and $\bigcup_{\alpha<\lambda} K_\alpha \in \mathbf{P}^\infty(X)$.*
3. *For all $C \in \prod_{P \in \mathbf{P}^\infty(X)} \mathscr{K}(P), \bigcup_{P \in \mathbf{P}^\infty(X)} (P \setminus C(P))$ has co-compact interior in X.*

Proof.

For the equivalence of 1 and 2: every path to infinity gives rise to a transfinite sequence and $P(f) \in \mathbf{P}^\infty(X)$ for any injective transfinite sequence f converging strictly to $\star$. For the equivalence of 1 and 3: use the equivalence of radiality and existence of closed spoke systems. Condition 3 is precisely saying that the sets generated from our spokes are indeed neighbourhoods (it is easily seen that they form a network). □

Corollary

If αX is radial at $\star$, then every closed, non-compact subset of X contains a path to infinity.

Theorem

If αX is radial at $\star$, then every compactification of X with finite remainder is radial at every point in the remainder.

Proof.

Suppose that γX is a finite compactification of X and note that $\alpha X \cong \gamma X/(\gamma X \setminus X)$. If $f : \lambda \to X$ doesn't contain a subsequence converging to a point in $\gamma X \setminus X$, then recursively pick neighbourhoods of those points missing a tail of a subsequence of f. Gluing these neighbourhoods together gives a neighbourhood of $\gamma X \setminus X$, which will give you the required contradiction. $\square$

Definition

A family $\mathscr{T} \subseteq [\omega]^{\aleph_0}$ is called a *tower* if it is well-ordered with respect to $\supseteq^*$. Such a tower is called *inextendible* if it has no infinite pseudointersection; that is, there is no infinite $P \subseteq \omega$ such that $P \subseteq^* T$ for all $T \in \mathscr{T}$. The tower number $\mathfrak{t}$ is the least size of an inextendible tower. Note that $\aleph_1 \leq \mathfrak{t} \leq \mathfrak{c}$.

Theorem

Assume αX is radial at $\star$ and let γX be a compactification of X such that either $\gamma X \setminus X$ is countable or X is countable and $|\gamma X| < \mathfrak{t}$. Then for every $A \subseteq X$ with non-compact closure, there exists a transfinite sequence in A that converges to some point in A. Moreover, if both X is sequential (= pseudoradial) and $\gamma X \setminus X$ is sequential / pseudoradial, then γX is sequential / pseudoradial.

Proof.

Use a similar argument as above for finite remainders. □

Now let X be a Stone space (compact, Hausdorff, 0-dimensional).

Definition

If $\mathscr{B}$ is a Boolean algebra and $\mathscr{U}$ is an ultrafilter on $\mathscr{B}$ (the set of these is denoted by $\mathbf{S}(\mathscr{B})$), then a subfilter $\mathscr{F} \subseteq \mathscr{U}$ is called a *lineariser* of $\mathscr{U}$ if $\mathscr{U}/\mathscr{F} = \{[u]_{\mathscr{F}} : u \in \mathscr{U}\}$ is *well-based* in $\mathscr{B}/\mathscr{F}$ (has a well-ordered neighbourhood base with respect to $\mathscr{B}/\mathscr{F}$). The collection of these is denoted $\mathbf{L}(\mathscr{U})$.

Definition

For a filter $\mathscr{F}$ on a Boolean algebra $\mathscr{B}$, define

$$C_{\mathscr{F}} := \{\mathscr{U} \in \mathbf{S}(\mathscr{B}) : \mathscr{F} \subseteq \mathscr{U}\}$$

Lemma

Let $\mathscr{B}$ be a Boolean algebra, $\mathscr{A} \subseteq \mathbf{S}(\mathscr{B})$ be given. Then $\overline{\mathscr{A}} = C_{\bigcap \mathscr{A}}$. In particular, $C_{\mathscr{F}}$ is closed for all filters $\mathscr{F}$ on $\mathscr{B}$.

Theorem

Let $\mathscr{B}$ be a Boolean algebra, $\mathscr{U} \in \mathbf{S}(\mathscr{B})$ be given. Then the following are equivalent:

1. $\mathbf{S}(\mathscr{B})$ *is radial at* $\mathscr{U}$.
2. $\forall \mathscr{A} \subseteq \mathbf{S}(\mathscr{B})$, *if* $\bigcap \mathscr{A} \subseteq \mathscr{U}$ *then there exists* $\mathscr{F} \in \mathbf{L}(\mathscr{U})$ *such that* $\bigcap(\mathscr{A} \uparrow \mathscr{F}) \subseteq \mathscr{U}$, *where* $\mathscr{A} \uparrow \mathscr{F} := \{\mathscr{V} \in \mathscr{A} : \mathscr{F} \subseteq \mathscr{V}\} = \mathscr{A} \cap C_{\mathscr{F}}$.
3. *For all* $B \in {}^{\mathbf{L}(\mathscr{U})}\mathscr{U}$, *there exists* $b \in \mathscr{B}$ *such that* $[b] \subseteq \bigcup_{\mathscr{F} \in \mathbf{L}(\mathscr{U})} C_{\mathscr{F}} \cap [B(\mathscr{F})]$.

Proof.

For the equivalence of 1 and 2: $(C_{\mathscr{F}})_{\mathscr{F} \in \mathbf{L}(\mathscr{U})}$ *is a spoke system when* $\mathscr{U}$ *is radial and use previous proof of radiality following from the existence of a spoke system. For the equivalence of 1 and 3: use the theorem for radiality at* $\star$ *together with* $\mathbf{S}(\mathscr{B}) \setminus \{\mathscr{U}\} \cong \mathbf{S}(\mathscr{B})$ *when* $\mathscr{U}$ *is not isolated (i.e. fixed).* $\square$