Playing with forcing

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Idealized forcings

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Examples

Classical examples are: Cohen forcing with the ideal of meager sets, random forcing with null sets, Miller with K_σ and Sacks with countable sets.

The generic real

A forcing notion of the form $\mathrm{Bor}(\omega^\omega)/I$ adds the *generic real*, denoted $\dot{g}$ and defined in the following way:

$$[\![\dot{g}(n) = m]\!] = [(n, m)]_I$$

where $[(n, m)]$ is the basic clopen in ω^ω).

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Genericity

Of course, the generic ultrafilter can be recovered from the generic real.

Properness

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If I is an ideal then the forcing notion $\mathbf{P}_I$ is proper if and only if for any $M \prec H_\kappa$ and any condition $B \in M \cap \mathbf{P}_I$

$$\{x \in B : x \text{ is generic over } M\} \notin I.$$

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$$\{x \in B : x \text{ is generic over } M\} \notin I.$$

Note that the set of generic reals over a countable model is always a Borel set.

Borel reading of names

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Theorem (Zapletal)

If the forcing $\mathbf{P}_I$ is proper and $\dot{x}$ is a name for a real then for each $B \in \mathbf{P}_I$ there is $C \leqslant B$ and a Borel function $f : C \longrightarrow \omega^\omega$ such that

$$C \Vdash \dot{x} = f(\dot{g}).$$

Examples

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Theorem (Zapletal)

If I is generated by closed sets then $\mathbf{P}_I$ has continuous reading of names.

Games

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"Banach-Mazur" games

Suppose for each Borel set $A \subseteq \omega^\omega$ $G(A)$ is a two player game in which Adam and Eve play natural numbers $x(i)$. Eve wins the game if $x \in P(A)$, where $P(A)$ is the payoff set for the game $G(A)$.

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Characterization of the ideals

We say that a game scheme as above describes ideal I if for each Borel $A \subseteq \omega^\omega$ $A \in I$ if and only if Eve has a winning strategy in $G(A)$.

Example

Let $\mathcal{E}$ denote the ideal generated by closed measure zero sets in 2^ω.

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A game

Consider the following game. The game is denoted by $G_\mathcal{E}$. It is played by Adam and Eve. In his n-th turn Adam picks $x_n \in 2^n$ so that $x_n \supseteq x_{n-1}$. In her n-th turn Eve picks a basic clopen $C_n \subseteq [x_n]$ such that its relative measure in $[x_n]$ is less than $1/n$. By the end of the game a sequence $x = \bigcup_n x_n \in 2^\omega$ is formed. Eve wins if

$$\text{either} \quad x \notin A \quad \text{or} \quad \forall^\infty n \; x \in C_n.$$

Otherwise Adam wins.

Proposition (MS)

For each $A \subseteq 2^\omega$ Eve has a winning strategy in $G_{\mathcal{E}}(A)$ if and only if $A \in \mathcal{E}$

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Proof

Suppose first that Eve has a winning strategy σ in the game $G_{\mathcal{E}}(A)$. For each $s \in 2^{<\omega}$ consider a partial play in which Adam picks $s{\restriction}k$ for $k \leqslant |s|$ and let C_s be the Eve's answer according to σ after this partial play. Put $E_n = \bigcup_{s \in 2^n} C_s$. Clearly E_n is a clopen set and $\mu(E_n) \leqslant 1/n$. Let $D_n = \bigcap_{m \geqslant n} E_n$. Now each D_n is a closed set of measure zero and since σ is a winning strategy we get that $A \subseteq \bigcup_n D_n$.

Proof — cntd

Conversely, assume that $A \in \mathcal{E}$. So there is a sequence D_n of closed sets of measure zero such that $A \subseteq \bigcup_n D_n$. Without loss of generality assume $D_n \subseteq D_{n+1}$. Let $T_n \subseteq \omega^{<\omega}$ be a tree such that $D_n = \lim T_n$. We define the strategy σ for Eve in the following way. After Adam picks $s \in 2^n$ in his n-th move consider the tree $(T_n)_s$. Since $\lim(T_n)_s$ is of measure zero there is $k < \omega$ such that

$$\frac{|(T_n)_s \cap 2^k|}{2^k} < \frac{1}{n}.$$

Let Eve's answer be the set $\bigcup_{t \in (T_n)_s \cap 2^k} [t]$. It is easy to check that this strategy is winning for Eve.

Axiom A

Recall that a forcing notion $\mathbf{P}$ satisfies Axiom A if there is a sequence of orderings $\leqslant_n$ on $\mathbf{P}$ such that $\leqslant_0 = \leqslant$, $\leqslant_{n+1} \subseteq \leqslant_n$ and

- if $\mathbf{P} \ni p_n, n < \omega$ are such that $p_{n+1} \leqslant_n p_n$ there is a $q \in \mathbf{P}$ such that $q \leqslant_n p_n$ for all n,
- for every $p \in \mathbf{P}$, for every n and for every ordinal name $\dot{x}$ there exist $\mathbf{P} \ni q \leqslant_n p$ and a countable set B such that $q \Vdash \dot{x} \in B$.

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Trees

Usually, Axiom A is present when the forcing has some tree reprezentation.

Proposition (MS)

The forcing $\mathbf{P}_{\mathcal{E}}$ satisfies Axiom A.

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Proof

For each $\mathcal{E}$-positive Borel set B there is a strategy σ for Adam in the game $G_{\mathcal{E}}(B)$. Such a strategy can be viewed as a tree T of partial plays in $G_{\mathcal{E}}(B)$ according to σ. Let $f_T : \lim T \to B$ be the function which assignes to a run t of the game $G_{\mathcal{E}}(B)$ the real x constructed by Adam in t. f_T is continuous and hence $A = \mathrm{rng}(f_T) \subseteq B$ is an analytic set. It is also $\mathcal{E}$-positive since the same strategy of Adam works for A.

Proof — cntd

Let T be a tree of a strategy for Adam in the game scheme $G_{\mathcal{E}}$. We will say that T is winning for Adam if for each $t \in \lim T$ we have $\exists^\infty n\; x \notin C_n$, where x and C_n are, respectively, the real constructed by Adam and the sequence of clopens picked by Eve in t. Obviously, such a T is a winning strategy for Adam in the game $G_{\mathcal{E}}(\mathrm{rng}(f_T))$. Hence the set $\mathrm{rng}(f_T)$ is $\mathcal{E}$-positive. Now, let $\mathbf{T}_{\mathcal{E}}$ be the forcing of trees winning for Adam in the game scheme $G_{\mathcal{E}}$. The ordering on $\mathbf{T}_{\mathcal{E}}$ is as follows: $T_0 \leqslant T_1$ if

$$\mathrm{rng}(f_{T_0}) \subseteq \mathrm{rng}(f_{T_1}).$$

Proof — cntd

By Solecki (Petruska) theorem any analytic set $A \subseteq \omega^\omega$ either contains an $\mathcal{E}$-positive $\mathbf{G}_\delta$ set or can be covered by an $\mathbf{F}_\sigma$ set in $\mathcal{E}$. Thus the forcing $\mathbf{P}_\mathcal{E}$ is dense in the forcing $\mathbf{Q}_\mathcal{E}$ of analytic $\mathcal{E}$-positive sets. It follows then that $\mathbf{T}_\mathcal{E} \ni T \mapsto \mathrm{rng}(f_T) \in \mathbf{Q}_E$ is a dense embedding. So what we get is that the three forcing notions $\mathbf{P}_\mathcal{E}$, $\mathbf{Q}_\mathcal{E}$ and $\mathbf{T}_\mathcal{E}$ are equivalent. We will establish Axiom A for $\mathbf{T}_\mathcal{E}$.

Proof — cntd

If $T \in \mathbf{T}_{\mathcal{E}}$ then each $t \in \lim T$ is a run of a game in which Adam wins. Pick $t \in \lim T$ and let x be the real constructed by Adam in t and C_n be the sequence of clopens constructed by Eve. We have that $\exists^\infty n\ x \notin C_n$. In particular there is the least such n_0 and since C_{n_0} is a clopen, there is the least $m_0 \geqslant n_0$ such that $[x{\restriction}m_0] \cap C_{n_0} = \emptyset$. Moreover, any $t' \in \lim T$ which contains $t{\restriction}m_0$ also has this property. If we pick for each $t \in \lim T$ such an $m_0(t) \in \omega$ then the family $\{t{\restriction}m_0(t) : t \in \lim T\}$ is an antichain and each $t \in \lim T$ extends one of its elements. We will call it the first front of the tree T and denote it by $F_1(T)$. Analogously we can define the n-th front of the tree T, $F_n(T)$. Note that

$$F_{n+1}(T) = \bigcup_{\tau \in F_n(T)} F_1(T_\tau).$$

Proof — cntd

Now we define fusion for $\mathbf{T}_{\mathcal{E}}$. Let $T \in \mathbf{T}_{\mathcal{E}}$ and $n \in \omega$, for each $\tau \in F_n(T)$ the set $\mathrm{rng}(f_{T_\tau})$ is still $\mathcal{E}$-positive. If we substitute for T_τ a tree of a winning strategy for Adam in the relativized game scheme $(G_{\mathcal{E}})_\tau$ we will still obtain a tree in $\mathbf{T}_{\mathcal{E}}$. The same is true after substitution for all T_τ for $\tau \in F_n(T)$. We define $\leqslant_n$ for $n < \omega$ as follows: $S \leqslant_n T$ if $S \supseteq F_n(T)$. Then if T_n is a fusion sequence, i.e. $T_{n+1} \leqslant_n T_n$ we have that $T = \bigcap_n T_n$ is a tree of a strategy for Adam and the strategy is winning because T contains infinitely many fronts. This ends the proof.

Yes, but...

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Answer:

Not much.

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Answer:

Not much.

Definition.

Say that an ideal I is generated by an analytic family of closed sets if there is a $\mathbf{\Sigma}^1_1$ subset $A \subseteq K(2^\omega)$ which generates I.

Theorem (MS)

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Proof

We begin with defining a game scheme G_I which is an “unfolded” version of a Banach-Mazur scheme, i.e. it detects whether $\pi[D] \in I$ for a closed $D \subseteq$. Pick a bijection $\rho : \omega \to \omega \times \omega$. By the theorem of Kechris, Louveau and Woodin $I \cap K(2^\omega) \in \mathbf{G}_\delta$, so let U_n be a sequence of open sets such that

$$I \cap K(2^\omega) = \bigcap_n U_n.$$

Let G_I be a game scheme in which Adam constructs an $x \in (2 \times \omega)^{\leqslant\omega}$ and Eve constructs a sequence E_n of closed sets in 2^ω.

Proof — cntd

In his n-th turn Adam can either define some next bits of x or decide to wait. In her n-th turn Eve picks a basic open set O_n in 2^ω such that if $n = \rho(k, l)$ then

$$2^\omega \setminus \bigcup_{i \leqslant l} O_{\rho(i,k)} \in D_l.$$

By the end of the game they have a sequence of closed set defined by

$$E_n = 2^\omega \setminus \bigcup_{i<\omega} O_{\rho(i,n)}.$$

Note that each $E_n \in I$. Adam wins the game $G_I(D)$ if

$$x \in D \quad \text{and} \quad \pi(x) \notin \bigcup_n E_n.$$

Lemma

If $D \subseteq$ is closed then Eve has a winning strategy in the game $G(D)$ if and only if

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Proof — cntd

Let $\mathbf{T}_I$ be the forcing of trees of a strategy for Adam that are winning. For $T_0, T_1 \in \mathbf{T}_I$ we define that $T_0 \leqslant T_1$ if

$$\pi[\lim T_0] \subseteq \pi[\lim T_1].$$

Now $T \mapsto \pi[\lim T]$ is a dense embedding from $\mathbf{T}_I$ to $\mathbf{Q}_I$ and the latter contains $\mathbf{P}_I$ as a dense subset by the Solecki theorem. Hence the forcings $\mathbf{T}_I$, $\mathbf{Q}_I$ and $\mathbf{P}_I$ are equivalent. We will establish Axiom A for $\mathbf{T}_I$.

Proof — cntd

For $T \in \mathbf{T}_I$ the winning condition says that in each game $t \in \lim T$ $\pi(x) \notin \bigcup_n E_n$. Using a compactness argument we get that for each n there is some k such that the partial play $t{\restriction}k$ already determines that $x \notin E_n$. This observation allows us to define the fronts $F_n(T)$ for each n and the rest of the proof follows the same lines as for $\mathbf{P}_{\mathcal{E}}$.

The end

Thank You.