Examples concerning iterated forcing II

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Outline

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1. On Suslin-free forcings, finishing the consistency of MA + $\neg$CH + There is no Kurepa tree

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2. The question of forcing chains of functions $f_\xi : \omega_1 \to \omega_1$ increasing modulo finite sets

Motivation: We will sketch the proof of the relative consistency (assuming the existence of a strongly inaccessible cardinal) of MA + $\neg$CH + There is no Kurepa tree

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2. And moreover for any c.c.c. forcing P of cardinality ω_1 $P \Vdash$ There is no Kurepa tree.

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1. First (using an inaccessible cardinal) obtain the consistency of CH + There is no Kurepa tree
2. And moreover for any c.c.c. forcing P of cardinality ω_1 $P \Vdash$ There is no Kurepa tree.
3. Assume: no c.c.c. forcing P of cardinality ω_1 forces that there is Kurepa tree

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2. Prove that if P is c.c.c. and adds an uncountable branch through an ω_1-tree, then there is Q which is c.c.c., does not add uncountable branches through ω_1-trees and

$$Q \Vdash \check{P} \text{ is not c.c.c.}$$

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3. Prove that if for each $\beta < \alpha$ we have $P_\beta \Vdash \dot{Q}_\beta$ does not add an uncountable branches through ω_1-trees, then P_α has this property as well as for each $\beta < \alpha$ we have that P_β forces that $P_{[\beta,\alpha)}$ has this property.

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Theorem

Suppose that A is a complete c.c.c. Boolean algebra and let T be a tree of height ω_1. If A^* adds a new branch through T, then A^* contains a reversed Souslin tree. In particular P^2 is not c.c.c.

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4. As $P \Vdash \dot{b} \neq \check{c}$ for any branch c of T, we conclude that $f[T']$ has height ω_1 and so is a Suslin tree.

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3. May w.l.o.g. assume that there are isomorphims $\pi_{\alpha,\beta} : a_\alpha \to a_\beta$ which lifts up to isomorphims of $(f_1^\alpha, ..., f_n^\alpha)$ and $(f_1^\beta, ..., f_n^\beta)$

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3. There is $Y \in u$ such that for $\alpha \in Y$ there are $t, s \in a$ such that

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4. If $\alpha_1, \alpha_2 \in Y$ and $\beta \in X_{\alpha_1} \cap X_{\alpha_1}$, then $\pi_{\alpha_1}(t), \pi_{\alpha_2}(t), \leq \pi_\beta(s)$ and so $\pi_{\alpha_1}(t), \pi_{\alpha_2}(t)$ are compatible, hence we get an uncountable branch through T, a contradiction.

Theorem

If P is c.c.c and adds a new branch through an ω_1-tree, then there is a c.c.c forcing Q that does not add a new branch through any ω_1-tree and

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(Devlin) It is consistent that there is no Kurepa tree and MA+$\neg$CH holds.

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(P.K.) "There is a countably tight compact space with no point of countable character" is consistent with MA+¬CH.
It is consistent that there is are compact spaces K, L and continuous onto map $f : K \to L$ such that K is first countable and L has no point of countable character.

Definition

Let $f, g : \omega_1 \to \omega_1$

$$\text{“} =_{f,g} \text{”} = \{\xi : f(\xi) = g(\xi)\}$$

$$\text{“} >_{f,g} \text{”} = \{\xi : f(\xi) > g(\xi)\}$$

We say that $f \leq^* g$ if and only if $>_{f,g}$ is finite and $=_{f,g}$ is co-uncountable. A $\leq^*$-chain is called strong chain.

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We say that $(X_\alpha : \alpha < \beta)$ is a strong chain of subsets of ω_1 iff for each $\alpha_1 < \alpha_2 < \beta$ we have

$$|X_{\alpha_1} \setminus X_{\alpha_2}| < \omega \ \ \& \ \ |X_{\alpha_2} \setminus X_{\alpha_1}| > \omega.$$

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Fact: The existence of a strong chain of functions $\omega_1 \to \omega_1$ of length κ is equivalent to the existence of a strong chain of subsets of ω_1 of length κ.

Theorem

CH or CC (Chang's Conjecture) imply that there are no strong chains. So there is no ZFC c.c.c. notion of forcing which adds a strong chain.

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Let $(X_\alpha : \alpha < \omega_2)$ be a strong chain of subsets of ω_1.

1. There is $\gamma < \omega_1$ such that $|\{X_\alpha \cap \gamma : \alpha \in \omega_2\}| = \omega_2$

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2. There is $C \subseteq \omega_2$, $|C| = \omega_2$ and $(\gamma_\xi)_{\xi<\omega_1}$ such that $X_\alpha \cap [\gamma_\xi, \gamma_{\xi+1}) \subset X_\beta \cap [\gamma_\xi, \gamma_{\xi+1})$ for all $\alpha < \beta$, $\alpha, \beta \in C$ and $\xi < \omega_1$

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3. CC implies that for any $c : [\omega_2]^2 \to \omega_1$ there is an uncountable $A \subseteq \omega_2$ and $\beta \in \omega_1$ such that $c[[A]^2] \subseteq \beta$.

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2. $\forall \alpha_1, \alpha_2 \in a_p \quad >_{f^p_{\alpha_1,\alpha_2}} \cap b_p \subseteq c(\alpha_1, \alpha_2)$
3. $p \leq q$ iff $a_p \supseteq a_q$, $b_p \supseteq b_q$, $f^p_\alpha \supseteq f^q_\alpha$ for $\alpha \in a_q$ and $\forall \alpha_1, \alpha_2 \in a_q \quad >_{f^p_{\alpha_1,\alpha_2}} \cap b_p = >_{f^q_{\alpha_1,\alpha_2}} \cap b_q$

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We will put $X_\alpha = \{\beta : f^p_\alpha(\beta) = 1,\ p \in G\}$ for a P-generic G.

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Theorem

(P.K.) It is consistent that there is a WCG Banach spaces where all operators are in the sequential closure of the linear span of projections from a projectional resolution of the identity

Definition

Let $f, g : \omega_1 \rightarrow \omega_1$

$$\geq_{f,g} = \{\xi : f(\xi) \geq g(\xi)\}$$

We say that $f \ll g$ if and only if $\geq_{f,g}$ is finite. A $\ll$-chain is called very strong chain.

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Theorem

(CH) there is no c.c.c. forcing which adds a very strong chain. So it cannot be added by an iteration of a σ-closed followed by a c.c.c. forcing.

Forcing by conditions $p = (a_p, b_p, F_p, A_p)$, where

1. $0 \in a_p \in [\omega_2]^{<\omega}$, $b_p \in [\omega_1]^{<\omega}$, $F_p = \{f_p^\alpha : \alpha \in a_p\}$, $A_p \in [\mathcal{F}]^{<\omega}$, and $f_p^\alpha : b_p \to \omega_1$, and for each $\beta \in b_p$ we have $f_p^0(\beta) = 0$

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2. $\forall \beta \in b_p \forall \alpha \in a_p \; f_p^\alpha(\beta) < \Phi(\beta)$
3. $\forall \beta \in b_p \forall \alpha_1 < \alpha_2; \alpha_1, \alpha_2 \in a_p$, if $d_{A_p,\beta}(\alpha_1, \alpha_2) \neq 0$, then

$$f_p^{\alpha_2}(\beta) \geq f_p^{\alpha_1}(\beta) + d_{A_p,\beta}(\alpha_1, \alpha_2)$$

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$$f_p^{\alpha_2}(\beta) \geq f_p^{\alpha_1}(\beta) + d_{A_p,\beta}(\alpha_1, \alpha_2)$$
4. $p \leq q$ iff $a_p \supseteq a_q$, $b_p \supseteq b_q$, $A_p \supseteq A_q$, $f_p^\alpha \supseteq f_q^\alpha$ for all $\alpha \in a_q$ and
5. $\forall \beta \in b_p - b_q \forall \alpha_1 < \alpha_2; \alpha_1, \alpha_2 \in a_q \quad f_p^{\alpha_2}(\beta) > f_p^{\alpha_1}(\beta)$

Theorem

(P.K.) It is consistent that here is a very strong chain of functions from ω_1 into ω_1 of length ω_2

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(C. Brech, P.K.) It is consistent that there is a compact hereditarily separable scattered compact space of Cantor-Bendixon height ω_2 and Cantor-Bendixon width ω.

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Theorem

(C. Brech, P.K.) It is consistent that there is a compact hereditarily separable scattered compact space of Cantor-Bendixon height ω_2 *and Cantor-Bendixon width* ω*. It is consistent that there is a Banach space of density* ω_2 *with no uncountable biorthogonal system.*