Examples concerning iterated forcing

Piotr Koszmider, piotr.koszmider@gmail.com

Motivation: We will sketch the proof of the relative consistency (assuming the existence of a strongly inaccessible cardinal) of MA + $\neg$CH + There is no Kurepa tree

1. MA = For every c.c.c. partial order P and a family $\mathcal{F}$ of cardinality $< 2^\omega$ of dense subsets of P there is a filter $G \subseteq P$ such that $D \cap G \neq \emptyset$ for all $D \in \mathcal{F}$

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered
2. $Lev_\alpha(T)$ is the set of such elements that $\{s \in T : s < t\} \equiv \alpha$

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered
2. $Lev_\alpha(T)$ is the set of such elements that $\{s \in T : s < t\} \equiv \alpha$
3. ω_1-tree = a tree with nonempty countable levels for $\alpha < \omega_1$ and $Lev_{\omega_1} = \emptyset$ i.e., with height ω_1

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered
2. $Lev_\alpha(T)$ is the set of such elements that $\{s \in T : s < t\} \equiv \alpha$
3. ω_1-tree = a tree with nonempty countable levels for $\alpha < \omega_1$ and $Lev_{\omega_1} = \emptyset$ i.e., with height ω_1
4. Branch through T = maximal linearly ordered subset of T

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered
2. $Lev_\alpha(T)$ is the set of such elements that $\{s \in T : s < t\} \equiv \alpha$
3. ω_1-tree = a tree with nonempty countable levels for $\alpha < \omega_1$ and $Lev_{\omega_1} = \emptyset$ i.e., with height ω_1
4. Branch through T = maximal linearly ordered subset of T
5. Antichain in T = set of pairwise incomparable elements

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered
2. $Lev_\alpha(T)$ is the set of such elements that $\{s \in T : s < t\} \equiv \alpha$
3. ω_1-tree = a tree with nonempty countable levels for $\alpha < \omega_1$ and $Lev_{\omega_1} = \emptyset$ i.e., with height ω_1
4. Branch through T = maximal linearly ordered subset of T
5. Antichain in T = set of pairwise incomparable elements
6. Suslin tree = ω_1-tree without uncountable antichain and without uncountable branch

1. Tree = partial order where for each $t \in T$ the set $\{s \in T : s < t\}$ is well ordered
2. $Lev_\alpha(T)$ is the set of such elements that $\{s \in T : s < t\} \equiv \alpha$
3. ω_1-tree = a tree with nonempty countable levels for $\alpha < \omega_1$ and $Lev_{\omega_1} = \emptyset$ i.e., with height ω_1
4. Branch through T = maximal linearly ordered subset of T
5. Antichain in T = set of pairwise incomparable elements
6. Suslin tree = ω_1-tree without uncountable antichain and without uncountable branch
7. Kurepa tree = ω_1-tree with more than ω_1 uncountable branches

Outline

Outline

1. On iterations of forcings

Outline

1. On iterations of forcings
2. On Suslin-free forcings

Outline

1. On iterations of forcings
2. On Suslin-free forcings
3. The consistency of MA + $\neg$CH + There is no Kurepa tree

1. Iterations of forcings of length α are sets of sequences of length α

1. Iterations of forcings of length α are sets of sequences of length α
2. Iterations of forcings P_0 of length 1 are just forcings

1. Iterations of forcings of length α are sets of sequences of length α
2. Iterations of forcings P_0 of length 1 are just forcings
3. If P_α is an iteration of length α and $\dot{Q}_\alpha$ is a P_α-name for an atomless partial order, then we define the iteration $P_\alpha * \dot{Q}_\alpha$ of length $\alpha + 1$

1. Iterations of forcings of length α are sets of sequences of length α
2. Iterations of forcings P_0 of length 1 are just forcings
3. If P_α is an iteration of length α and $\dot{Q}_\alpha$ is a P_α-name for an atomless partial order, then we define the iteration $P_\alpha * \dot{Q}_\alpha$ of length $\alpha + 1$
4. $p^\frown \dot{q} \in P_\alpha * \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \Vdash \dot{q} \in \dot{Q}_\alpha$

1. Iterations of forcings of length α are sets of sequences of length α
2. Iterations of forcings P_0 of length 1 are just forcings
3. If P_α is an iteration of length α and $\dot{Q}_\alpha$ is a P_α-name for an atomless partial order, then we define the iteration $P_\alpha * \dot{Q}_\alpha$ of length $\alpha + 1$
4. $p^\frown \dot{q} \in P_\alpha * \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \Vdash \dot{q} \in \dot{Q}_\alpha$
5. $p^\frown \dot{q} \leq p'^\frown \dot{q}'$ iff $p \leq_{P_\alpha} p'$ and $p \Vdash \dot{q} \leq_{\dot{Q}_\alpha} \dot{q}'$

1. Iterations of forcings of length α are sets of sequences of length α
2. Iterations of forcings P_0 of length 1 are just forcings
3. If P_α is an iteration of length α and $\dot{Q}_\alpha$ is a P_α-name for an atomless partial order, then we define the iteration $P_\alpha * \dot{Q}_\alpha$ of length $\alpha + 1$
4. $p^\frown \dot{q} \in P_\alpha * \dot{Q}_\alpha$ iff $p \in P_\alpha$ and $p \Vdash \dot{q} \in \dot{Q}_\alpha$
5. $p^\frown \dot{q} \leq p'^\frown \dot{q}'$ iff $p \leq_{P_\alpha} p'$ and $p \Vdash \dot{q} \leq_{\dot{Q}_\alpha} \dot{q}'$
6. If $P_{\alpha'}$s are iterations of lengths α' respectively and $P_{\alpha'}|\alpha'' = P_{\alpha''}$ for all $\alpha'' < \alpha' < \alpha$ then we define the iteration P_α of length α with supports $< \kappa$:

$$p \in P_\alpha \text{ iff } \forall \alpha' < \alpha \ \ p|\alpha' \in P_{\alpha'}$$

$$supp(p) = \{\alpha' < \alpha : p(\alpha') \neq 1_{\dot{Q}_\alpha}\} \text{ has cardinality } < \kappa$$

Observations A:

1. We identify P_β for $\beta < \alpha$ with a suborder of P_α. Also P_β-names correspond to some P_α-names.

Observations A:

1. We identify P_β for $\beta < \alpha$ with a suborder of P_α. Also P_β-names correspond to some P_α-names.
2. For each $\beta < \alpha$ the iteration P_α is equivalent to $P_\beta * P_{[\alpha,\beta)}$ where $\dot{P}_{[\alpha,\beta)}$ is an appropriate iteration

Observations A:

1. We identify P_β for $\beta < \alpha$ with a suborder of P_α. Also P_β-names correspond to some P_α-names.
2. For each $\beta < \alpha$ the iteration P_α is equivalent to $P_\beta * P_{[\alpha,\beta)}$ where $\dot{P}_{[\alpha,\beta)}$ is an appropriate iteration
3. If D is dense in P_β then P_α forces that

$$\dot{G}|\beta = \{p|\beta : p \in \dot{G}\} \cap \check{D} \neq \emptyset$$

Observations A:

1. We identify P_β for $\beta < \alpha$ with a suborder of P_α. Also P_β-names correspond to some P_α-names.
2. For each $\beta < \alpha$ the iteration P_α is equivalent to $P_\beta {}^* P_{[\alpha,\beta)}$ where $\dot{P}_{[\alpha,\beta)}$ is an appropriate iteration
3. If D is dense in P_β then P_α forces that

$$\dot{G}|\beta = \{p|\beta : p \in \dot{G}\} \cap \check{D} \neq \emptyset$$

4. If $\dot{D}$ is a P_β-name for a dense subset of $\dot{Q}_\beta$, then P_α forces that

$$\dot{G}(\beta) = \{p(\beta) : p \in \dot{G}\} \cap \dot{D} \neq \emptyset$$

Observations B:

1. If $P_\beta \Vdash \dot{Q}_\beta$ is c.c.c for each $\beta < \alpha$, and P_α is an iteration with finite support, then P_α is c.c.c.

Observations B:

1. If $P_\beta \Vdash \dot{Q}_\beta$ is c.c.c for each $\beta < \alpha$, and P_α is an iteration with finite support, then P_α is c.c.c.
2. But there could be P_1, Q_1 both c.c.c. such that $P_1 * \check{Q}_1$ is not c.c.c. (because $P_1 \not\Vdash \check{Q}_1$ is c.c.c.)

Observations B:

1. If $P_\beta \Vdash \dot{Q}_\beta$ is c.c.c for each $\beta < \alpha$, and P_α is an iteration with finite support, then P_α is c.c.c.
2. But there could be P_1, Q_1 both c.c.c. such that $P_1{}^*\check{Q}_1$ is not c.c.c. (because $P_1 \not\Vdash \check{Q}_1$ is c.c.c.)
3. If P is reversed Suslin tree then P is c.c.c. but $P^*\check{P}$ is not c.c.c. because $P \times P \subseteq P^*\check{P}$ is not c.c.c.

Observations C:

1. In general if $\dot{x}$ is a P_α-name for α a limit ordinal of (large) cofinality there may not be $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Observations C:

1. In general if $\dot{x}$ is a P_α-name for α a limit ordinal of (large) cofinality there may not be $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$
2. Let κ be a cardinal. Let P_α be an iteration with finite supports of c.c.c. forcings where $\kappa < cf(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Theorem

Let κ be a cardinal. Let P_α be an iteration with finite supports of c.c.c. forcings where $\kappa < cf(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Theorem

Let κ be a cardinal. Let P_α be an iteration with finite supports of c.c.c. forcings where $\kappa < cf(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Proof.

Theorem

Let κ be a cardinal. Let P_α be an iteration with finite supports of c.c.c. forcings where $\kappa < cf(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Proof.

1. If $A \subseteq P_\alpha$ is an antichain, then $\bigcup\{supp(p) : p \in A\}$ is bounded in α.

□

Theorem

Let κ be a cardinal. Let P_α be an iteration with finite supports of c.c.c. forcings where $\kappa < cf(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Proof.

1. If $A \subseteq P_\alpha$ is an antichain, then $\bigcup\{supp(p) : p \in A\}$ is bounded in α.
2. For every $\xi < \kappa$ define a maximal antichain A_ξ among conditions of P_α which force $\check{\xi} \in \dot{x}$

□

Theorem

Let κ be a cardinal. Let P_α be an iteration with finite supports of c.c.c. forcings where $\kappa < cf(\alpha)$ is uncountable. If $P_\alpha \Vdash \dot{x} \subseteq \check{\kappa}$. Then there is $\beta < \alpha$ and a P_β-name $\dot{y}$ such that $P_\alpha \Vdash \dot{x} = \dot{y}$

Proof.

1. If $A \subseteq P_\alpha$ is an antichain, then $\bigcup\{supp(p) : p \in A\}$ is bounded in α.
2. For every $\xi < \kappa$ define a maximal antichain A_ξ among conditions of P_α which force $\check{\xi} \in \dot{x}$
3. Define $\dot{y} = \bigcup_{\xi\in\kappa}\{\check{\xi}\} \times A_\xi$

□

Theorem

(GCH) There is a finite support iteration P_κ of length ω_2 of c.c.c. forcings such that $P_{\omega_2} \Vdash MA + 2^\omega = \omega_2$

Theorem

(GCH) There is a finite support iteration P_κ of length ω_2 of c.c.c. forcings such that $P_{\omega_2} \Vdash MA + 2^\omega = \omega_2$

Proof.

Theorem

(GCH) There is a finite support iteration P_κ of length ω_2 of c.c.c. forcings such that $P_{\omega_2} \Vdash MA + 2^\omega = \omega_2$

Proof.

Do the right book-keeping so that

Theorem

(GCH) There is a finite support iteration P_κ of length ω_2 of c.c.c. forcings such that $P_{\omega_2} \Vdash MA + 2^\omega = \omega_2$

Proof.

Do the right book-keeping so that whenever $P_{\omega_2} \Vdash |\dot{P}| \leq \omega_1$ and $P_{\omega_2} \Vdash \dot{P}$ is c.c.c., and $\{\dot{D}_\xi : \xi < \omega_1\}$ are P_{ω_2}-names for dense sets of $\dot{P}$ then there is $\beta < \omega_2$ such that $P_\beta \Vdash \dot{P} = \dot{Q}_\beta$ and there are P_β-names $\{\dot{E}_\xi : \xi < \omega_1\}$ such that $P_\beta \Vdash \dot{E}_\xi = \dot{D}_\xi$ for $\xi < \omega_1$

Theorem

(GCH) There is a finite support iteration P_κ of length ω_2 of c.c.c. forcings such that $P_{\omega_2} \Vdash MA + 2^\omega = \omega_2$

Proof.

Do the right book-keeping so that whenever $P_{\omega_2} \Vdash |\dot{P}| \leq \omega_1$ and $P_{\omega_2} \Vdash \dot{P}$ is c.c.c., and $\{\dot{D}_\xi : \xi < \omega_1\}$ are P_{ω_2}-names for dense sets of $\dot{P}$ then there is $\beta < \omega_2$ such that $P_\beta \Vdash \dot{P} = \dot{Q}_\beta$ and there are P_β-names $\{\dot{E}_\xi : \xi < \omega_1\}$ such that $P_\beta \Vdash \dot{E}_\xi = \dot{D}_\xi$ for $\xi < \omega_1$
Then P_β forces that $\dot{Q}_\beta$ forces that

$$\dot{G}(\beta) = \{p(\beta) : p \in \dot{G}\}$$

is a filter in $\dot{P} = \dot{Q}_\beta$ meeting all $\dot{E}_\xi = \dot{D}_\xi$.

Theorem

(GCH) There is a finite support iteration P_κ of length ω_2 of c.c.c. forcings such that $P_{\omega_2} \Vdash MA + 2^\omega = \omega_2$

Proof.

Do the right book-keeping so that whenever $P_{\omega_2} \Vdash |\dot{P}| \leq \omega_1$ and $P_{\omega_2} \Vdash \dot{P}$ is c.c.c., and $\{\dot{D}_\xi : \xi < \omega_1\}$ are P_{ω_2}-names for dense sets of $\dot{P}$ then there is $\beta < \omega_2$ such that $P_\beta \Vdash \dot{P} = \dot{Q}_\beta$ and there are P_β-names $\{\dot{E}_\xi : \xi < \omega_1\}$ such that $P_\beta \Vdash \dot{E}_\xi = \dot{D}_\xi$ for $\xi < \omega_1$
Then P_β forces that $\dot{Q}_\beta$ forces that

$$\dot{G}(\beta) = \{p(\beta) : p \in \dot{G}\}$$

is a filter in $\dot{P} = \dot{Q}_\beta$ meeting all $\dot{E}_\xi = \dot{D}_\xi$. This is preserved from P_β to P_{ω_2} because P_{ω_2} is equivalent to $P_\beta {}^* P_{[\beta,\omega_2)}$

□

Motivation: We will sketch the proof of the relative consistency (assuming the existence of a strongly inaccessible cardinal) of MA + $\neg$CH + There is no Kurepa tree

Proof.

Preparatory stage

Proof.

Preparatory stage

1. First (using an inaccessible cardinal) obtain the consistency of CH + There is no Kurepa tree

□

Proof.

Preparatory stage

1. First (using an inaccessible cardinal) obtain the consistency of CH + There is no Kurepa tree
2. And moreover for any c.c.c. forcing P of cardinality ω_1 $P \Vdash$ There is no Kurepa tree.

□

Proof.

Preparatory stage

1. First (using an inaccessible cardinal) obtain the consistency of CH + There is no Kurepa tree
2. And moreover for any c.c.c. forcing P of cardinality ω_1 $P \Vdash$ There is no Kurepa tree.
3. Assume: no c.c.c. forcing P of cardinality ω_1 forces that there is Kurepa tree

□

Proof.

Main stage

Proof.

Main stage

1. Iterate all c.c.c forcings of cardinality ω_1 which do not add uncountable branches through an ω_1-trees

$\square$

Proof.

Main stage

1. Iterate all c.c.c forcings of cardinality ω_1 which do not add uncountable branches through an ω_1-trees
2. Prove that if P is c.c.c. and adds an uncountable branch through an ω_1-tree, then there is Q which is c.c.c., does not add uncountable branches through ω_1-trees and

$$Q \Vdash \check{P} \text{ is not c.c.c.}$$

□

Proof.

Main stage

1. Iterate all c.c.c forcings of cardinality ω_1 which do not add uncountable branches through an ω_1-trees
2. Prove that if P is c.c.c. and adds an uncountable branch through an ω_1-tree, then there is Q which is c.c.c., does not add uncountable branches through ω_1-trees and

$$Q \Vdash \check{P} \text{ is not c.c.c.}$$

3. Prove that if for each $\beta < \alpha$ we have $P_\beta \Vdash \dot{Q}_\beta$ does not add an uncountable branches through ω_1-trees, then P_α has this property as well as for each $\beta < \alpha$ we have that P_β forces that $P_{[\beta,\alpha)}$ has this property.

□