Seven characterizations of non-meager P-filters

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A speed-of-light introduction to P-points

Every filter $\mathcal{F}$ is on ω and $\{\text{cofinite sets}\} \subseteq \mathcal{F} \subsetneq \mathcal{P}(\omega)$.

Definition

- A filter $\mathcal{F}$ a P*-filter* if every countable $\mathcal{X} \subseteq \mathcal{F}$ has a pseudointersection in $\mathcal{F}$.
- An ultrafilter $\mathcal{U}$ that is a P-filter is called a P*-point*.

Theorem (W. Rudin, 1956)

Assume CH. *Then there are* P*-points.*

It is not hard to see that CH can be weakened to $\mathfrak{d} = \mathfrak{c}$.

Proposition (folklore)

*There are non-*P*-points.*

Theorem (Shelah, 1982)

It is consistent that there are no P*-points.*

Countable dense homogeneity

All spaces are assumed to be Hausdorff and separable.

Definition (Bennett, 1972)

A space X is *countable dense homogeneous* (briefly, CDH) if for every pair (D, E) of countable dense subsets of X there exists a homeomorphism $h : X \longrightarrow X$ such that $h[D] = E$.

Examples:

- $\mathbb{R}$ (Cantor, 1895), $\mathbb{R}^n$ (Brouwer, 1913).
- Any euclidean manifold of weight less than $\mathfrak{b}$ (Steprāns and Zhou, 1988).
- The Hilbert cube $[0, 1]^\omega$ (Fort, 1962).
- Under MA(σ-centered), there exists a homogeneous CDH Bernstein set $X \subseteq 2^\omega$ (Baldwin and Beaudoin, 1989).

Non-examples:

- $\mathbb{Q}$ (Trivial!), $\mathbb{Q}^\omega$ (Fitzpatrick and Zhou, 1992).

What about infinite powers?

Notice that taking powers adds new homeomorphisms. (If nothing else, permutations of the coordinates...)

Question (Fitzpatrick and Zhou, 1990)

Which $X \subseteq 2^\omega$ are such that X^ω is homogeneous? Countable dense homogeneous?

The first question has a very clear answer, namely

"All of them. (And many more!)"

The following remarkable theorem is based on work of Lawrence and Motorov.

Theorem (Dow and Pearl, 1997)

If X is first-countable and zero-dimensional then X^ω is homogeneous.

A partial answer (for Borel sets)

From now on, all spaces will be separable metrizable.

Theorem (Hrušák and Zamora Avilés, 2005)

For a Borel $X \subseteq 2^\omega$, the following conditions are equivalent.

- *X^ω is* CDH.
- *X is a G_δ (equivalently, Polish).*

At this point, it is natural to conjecture whether being a G_δ is in fact the characterizing property that we're looking for...

Question (Hrušák and Zamora Avilés, 2005)

Is there a non-G_δ subset X of 2^ω such that X^ω is CDH*?*

Given Baldwin and Beaudoin's result, a Bernstein set looks like a natural candidate. Unfortunately...

Theorem (Hernández-Gutiérrez, 2013)

If X is crowded and X^ω is CDH*, then X contains a copy of 2^ω.*

What about an ultrafilter?

Notice that, whenever $\mathcal{X} \subseteq \mathcal{P}(\omega)$, it is possible to view $\mathcal{X}$ as a subset of 2^ω through characteristic functions. In particular, $\mathcal{X}$ gets the subspace topology.

From now on, we will focus on the case when $\mathcal{X} = \mathcal{F}$ is a filter.

Proposition (folklore)

Let $\mathcal{F}$ be an ultrafilter (or even just a non-meager filter). Then $\mathcal{F}$ has the following properties.

- *$\mathcal{F}$ does not have the property of Baire.*
- *$\mathcal{F}$ is not analytic or co-analytic (hence not G_δ!)*
- *$\mathcal{F}$ is a topological group (hence homogeneous).*
- *$\mathcal{F}$ is a Baire space.*
- *$\mathcal{F}$ contains many homeomorphic copies of 2^ω (take $z \uparrow$ for any coinfinite $z \in \mathcal{F}$).*

Theorem (Medini and Milovich, 2012)

Assume MA(countable)*.*

- *There exists an ultrafilter* $\mathcal{U}$ *such that* $\mathcal{U}$ *is* CDH*.*
- *There exists an ultrafilter* $\mathcal{U}$ *such that* $\mathcal{U}^\omega$ *is* CDH*.*

The following question is a very specific instance of a **very general theme:** do combinatorial constraints on the family $\mathcal{X} \subseteq \mathcal{P}(\omega)$ have topological consequences/equivalents on the space $\mathcal{X} \subseteq 2^\omega$?

Question (Medini and Milovich, 2012)

Is every P*-point necessarily* CDH*?*

Theorem (Hernández-Gutiérrez and Hrušák, 2013)

Let $\mathcal{F}$ *be a non-meager* P*-filter. Then* $\mathcal{F}$ *and* $\mathcal{F}^\omega$ *are* CDH*.*

Great! ☺ Do non-meager P-filters exist?

It's a long-standing open problem... ☹

Question (Just, Mathias, Prikry and Simon, 1990)

Can we construct in ZFC *a non-meager* P*-filter?*

Theorem (Just, Mathias, Prikry and Simon, 1990)

If one of the following assumptions holds, then there exists a non-meager P*-filter.*

- $\mathfrak{t} = \mathfrak{b}$.
- $\mathfrak{b} < \mathfrak{d}$.
- $\mathrm{cof}([\mathfrak{d}]^\omega, \subseteq) \leq \mathfrak{d}$.

The last condition is particularly interesting because its negation has large cardinal strength.

Marciszewski's result

The following topological characterization was already known.

Definition

A space X is *completely Baire* (briefly, CB) if every closed subspace of X is Baire.

Theorem (Marciszewski, 1998)

For a filter $\mathcal{F}$, the following are equivalent.

- *$\mathcal{F}$ is a non-meager* P*-filter.*
- *$\mathcal{F}$ is completely Baire.*

One direction is a very cute tiny little elegant proof... ☺

The other direction is ugly. ☹

(Note: This might just mean that I don't understand it...)

But it doesn't matter because we will give a new, more systematic proof! ☺

Can we at least construct a non-CDH ultrafilter in ZFC?

Theorem (Medini and Milovich, 2012)

Assume MA(countable). *Then there exists an ultrafilter that is not* CDH.

Theorem (Repovš, Zdomskyy and Zhang, to appear)

In ZFC, *there exists a non-meager filter that is not* CDH.

The next part of the talk will be about the proof of the following characterization. At the same time we will obtain five more characterizations. (Recall that $5 + 2 = 7$.)

Theorem

For a filter $\mathcal{F}$, the following are equivalent.

- *$\mathcal{F}$ is a non-meager* P*-filter.*
- *$\mathcal{F}$ is countable dense homogeneous.*

Strengthening a result of Miller

Miller showed that P-points are preserved under Miller forcing, then non-chalantly remarked that his proof can be modified to obtain the following result.

Definition

A filter $\mathcal{F}$ has the *strong Miller property* if for every countable crowded $Q \subseteq \mathcal{F}$ there exists a crowded $Q' \subseteq Q$ such that $Q' \subseteq z\uparrow$ for some $z \in \mathcal{F}$.

Theorem (Miller, 1983)

Every P*-point has the strong Miller property.*

Miller's result can be generalized as follows, yielding the **combinatorial core of our proof.**

Theorem

Every non-meager P*-filter has the strong Miller property.*

Actually, the proof gives something stronger.

Lemma

Let $\mathcal{F}$ be a non-meager (not necessarily P*-)filter. If $Q \subseteq \mathcal{F}$ is countable, crowded and has a pseudointersection in $\mathcal{F}$, then there exists a crowded $Q' \subseteq Q$ such that $Q' \subseteq z\uparrow$ for some $z \in \mathcal{F}$.*

Notice that the strong Miller property implies the following purely topological property.

Definition

A space X has the *Miller property* if for every countable crowded $Q \subseteq X$ there exists a crowded $Q' \subseteq Q$ with compact closure.

The Cantor-Bendixson property

Before stating the main theorem, we need two more definitions.

Definition

A space X has the *Cantor-Bendixson property* if every closed subset of X is either scattered or contains a copy of 2^ω.

Notice that the Cantor-Bendixson property is intermediate between the Miller property and being completely Baire.
Just like we did for the Miller property, if $X = \mathcal{F}$ is a filter, we can consider a strong version of this property.

Definition

A filter $\mathcal{F}$ has the *strong Cantor-Bendixson property* if every closed subset of $\mathcal{F}$ is either scattered or contains a copy K of 2^ω such that $K \subseteq z\uparrow$ for some $z \in \mathcal{F}$.

The seven characterizations (finally!)

Theorem

For a filter $\mathcal{F}$, the following are equivalent.

1. *$\mathcal{F}$ is a non-meager P-filter.*
2. *$\mathcal{F}$ has the strong Miller property.*
3. *$\mathcal{F}$ has the Miller property.*
4. *$\mathcal{F}$ has the strong Cantor-Bendixson property.*
5. *$\mathcal{F}$ has the Cantor-Bendixson property.*
6. *$\mathcal{F}$ is completely Baire.*
7. *$\mathcal{F}$ is relatively countable dense homogeneous in 2^ω.*
8. *$\mathcal{F}$ is countable dense homogeneous.*

First, prove that $(1), \ldots, (6)$ are equivalent, using the proof of Marciszewski for $(6) \to (1)$. Then, $(1) \to (7)$ follows from the proof of Hernández-Gutiérrez and Hrušák, $(7) \to (8)$ is trivial, and the proof of $(8) \to (3)$ is in the next slide.

CDH filters have the Miller property

Assume that $\mathcal{F}$ is countable dense homogeneous. In particular, by a future slide (that you might never see), $\mathcal{F}$ must be non-meager.

Let $E = \{\text{cofinite sets}\}$. Notice that *every* subset of E has a pseudointersection in $\mathcal{F}$. (Just take ω!)

Now let $Q \subseteq \mathcal{F}$ be countable and crowded. Extend Q to a countable dense subset D of $\mathcal{F}$.

Let $h : \mathcal{F} \longrightarrow \mathcal{F}$ be a homeomorphism such that $h[D] = E$.

Since $\mathcal{F}$ is non-meager, the lemma shows that $R = h[Q]$ has a crowded subset R' with compact closure.

Since h is a homeomorphism, it follows that $Q' = h^{-1}[R']$ is a crowded subset of Q with compact closure. Therefore $\mathcal{F}$ has the Miller property.

Types of countable dense subsets

Given D, E countable dense subsets of X, define $D \sim_X E$ if there exists $h \in \mathcal{H}(X)$ such that $h[D] = E$.

Definition

The *number of types of countable dense subsets of X* (briefly, $\mathrm{TCD}(X)$) is the number of equivalence classes of $\sim_X$.

Clearly, a space X is CDH if and only if $\mathrm{TCD}(X) = 1$.
Also notice that $\mathrm{TCD}(X) \leq \mathfrak{c}$ for every space X.
The following is essentially due to Hrušák and Van Mill (2013).

Theorem

Suppose that X is not completely Baire but has a completely Baire dense subset. Then $\mathrm{TCD}(X) = \mathfrak{c}$.

Using this result, we will show that it is consistent that $\mathrm{TCD}(\mathcal{F}) \in \{1, \mathfrak{c}\}$ for every filter $\mathcal{F}$.

Meager filters have $\mathfrak{c}$ types of countable dense subsets

Definition

A space X has the *perfect set property for open sets* (briefly, PSP(open)) if every open subset of X is either countable or it contains a copy of 2^ω.

By Luzin's theorem, every analytic space has PSP(open).
Also, every filter has the PSP(open).
The following is essentially due to Hrušák and Van Mill (2013).

Theorem

Suppose that X is not Baire but has PSP(open)*. Then* $\mathrm{TCD}(X) = \mathfrak{c}$*.*

Corollary

Let $\mathcal{F}$ be a meager filter. Then $\mathrm{TCD}(\mathcal{F}) = \mathfrak{c}$*.*

Two consistent characterizations

Given a space $X \subseteq 2^\omega$, let RTCD(X) be the number of *relative* types of countable dense subsets of X in 2^ω.

Theorem

Assume $\mathfrak{u} < \mathfrak{g}$. For a filter $\mathcal{F}$, the following are equivalent.

1. *$\mathcal{F}$ is a non-meager* P*-filter.*
2. TCD($\mathcal{F}$) $< \mathfrak{c}$.
3. RTCD($\mathcal{F}$) $< \mathfrak{c}$.

We already know that (1) $\rightarrow$ (3), and (3) $\rightarrow$ (2) is obvious.
To show that (2) $\rightarrow$ (1), assume that that $\mathcal{F}$ is meager or a non-P-filter. We will show that TCD($\mathcal{F}$) $= \mathfrak{c}$.
If $\mathcal{F}$ is meager, we already known that TCD($\mathcal{F}$) $= \mathfrak{c}$. So we can assume that $\mathcal{F}$ is a non-meager non-P-filter.
In particular, $\mathcal{F}$ is not CB. So it would be enough to show that $\mathcal{F}$ has a CB dense subset...

How do we get a CB dense subset?

Question

Does every non-meager filter have a CB *dense subset?*

Consistently, yes. Actually, something stronger is true.

Theorem

Assume $\mathfrak{u} < \mathfrak{g}$*. Then every non-meager filter has a non-meager* P*-subfilter.*

To prove the above theorem, we will make use of some 'coherence of filters' voodoo:

Theorem (see the article of Blass in the Handbook)

Assume $\mathfrak{u} < \mathfrak{g}$*. The there exists a* P*-point* $\mathcal{U}$ *such that for every non-meager filter* $\mathcal{F}$ *there exists a finite-to-one function* $f : \omega \longrightarrow \omega$ *such that* $f(\mathcal{U}) = f(\mathcal{F})$*.*

Recall that, given $f : \omega \longrightarrow \omega$ and $\mathcal{X} \subseteq \mathcal{P}(\omega)$,

$$f(\mathcal{X}) = \{x \subseteq \omega : f^{-1}[x] \in \mathcal{X}\}.$$

If f is finite-to-one, it is easy to see that $f(\mathcal{F})$ is a filter (resp. ultrafilter) whenever $\mathcal{F}$ is a filter (resp. ultrafilter).
Fix a non-meager filter $\mathcal{F}$. Get a finite-to-one $f : \omega \longrightarrow \omega$ such that $f(\mathcal{U}) = f(\mathcal{F})$, where $\mathcal{U}$ is the P-point given by the theorem.
It is clear that $\langle\{f^{-1}[x] : x \in f(\mathcal{F})\}\rangle = \langle\{f^{-1}[x] : x \in f(\mathcal{U})\}\rangle$ is a subfilter of $\mathcal{F}$.
By the following proposition, it is a non-meager P-filter.

Proposition

Let $\mathcal{F}$ be a filter and let $f : \omega \longrightarrow \omega$ be finite-to-one.

1. *If $\mathcal{F}$ is a* P*-filter, then $f(\mathcal{F})$ is a* P*-filter.*
2. *If $\mathcal{F}$ is a* P*-filter, then $\langle\{f^{-1}[x] : x \in \mathcal{F}\}\rangle$ is a* P*-filter.*
3. *If $\mathcal{F}$ is non-meager, then $\langle\{f^{-1}[x] : x \in \mathcal{F}\}\rangle$ is non-meager.*

Does that hold in ZFC?

It depends on what you mean by 'that'...

If 'that' $=$ 'every non-meager filter has a non-meager P-subfilter', then no. It is actually independent of ZFC.

Theorem

Assume $\diamondsuit$. Then there exists an ultrafilter $\mathcal{U}$ such that whenever $\mathcal{X} \subseteq \mathcal{U}$, one of the following holds.

1. *$\mathcal{X}$ has a countable subset with no pseudointersection in $\mathcal{U}$. (In particular, $\mathcal{X}$ is not a P-filter.)*
2. *$\mathcal{X}$ has a pseudointersection. (In particular, $\mathcal{X}$ is meager.)*

Whenever you prove something using $\diamondsuit$, there's a natural question to ask:

Question

Can we weaken to CH the assumption of $\diamondsuit$?

If ‘that’ $=$ ‘every non-meager filter has a CB dense subset’, then we don’t know. Actually we don’t even know the answer to the following question.

Question

Can we construct in ZFC *a (non-meager) filter that is not* CB *but has a* CB *dense subset?*

The following question goes in the ‘opposite direction’.

Question

For a filter $\mathcal{F}$*, is having a* CB *dense subset equivalent to having a non-meager* P*-subfilter?*

Question

Let D *be a* CB *dense subset of a filter* $\mathcal{F}$*. Is* $\langle D \rangle$ *necessarily a (non-meager)* P*-filter?*

If ‘that’ $=$ ‘the characterization involving $\mathrm{TCD}(\mathcal{F})$’, then we also don’t know. It is equivalent to the following question.

Question

Is it consistent that there exists a filter $\mathcal{F}$ such that $1 < \mathrm{TCD}(\mathcal{F}) < \mathfrak{c}$?

The following is also open.

Question

Can we construct in ZFC *a non-meager filter $\mathcal{F}$ such that $\mathrm{TCD}(\mathcal{F}) = \mathfrak{c}$?*

We know that such a filter exists under $\mathfrak{u} < \mathfrak{g}$. Furthermore:

Theorem

Assume MA(countable)*. Then there exists an ultrafilter $\mathcal{U}$ such that $\mathrm{TCD}(\mathcal{U}) = \mathfrak{c}$.*

What happens in arbitrary spaces?

Notice that some of the properties that we considered are purely topological. So it is natural to ask whether they are equivalent for arbitrary spaces.

Theorem

For a co-analytic space X, the following are equivalent.

1. *X is Polish.*
2. *X has the Miller property.*
3. *X has the Cantor-Bendixson property.*
4. *X is completely Baire.*

Notice that $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4)$ holds for arbitrary spaces.

Theorem

There exists a ZFC *counterexample to the implication $(i) \leftarrow (i+1)$ for every $i = 1, 2, 3$.*

The case $i = 2$ is based on an example of Brendle.

Optimizing the definability

It is easy to see that, under Projective Determinacy, the equivalence result extends to all projective spaces.
So the following is the best we can hope for:

Question

Is there a consistent analytic counterexample to $(i) \leftarrow (i+1)$ *for every* $i = 1, 2, 3$*?*

In the case $i = 1$, we know that the answer is yes.
Recall that λ'-sets of size ω_1 exist in ZFC.

Proposition

Let $Y \subseteq 2^\omega$ *be a* λ'*-set. Then* $X = 2^\omega \setminus Y$ *has the Miller property but it is not Polish. Furthermore, if* $|Y| = \omega_1$ *and* $\mathrm{MA} + \neg\mathrm{CH} + \omega_1 = \omega_1^{\mathrm{L}}$ *holds, the space* X *is analytic.*

In fact, under $\mathrm{MA} + \neg\mathrm{CH} + \omega_1 = \omega_1^{\mathrm{L}}$, every set of size ω_1 is co-analytic (this is a classical result of Martin and Solovay).