Transversal and independent topologies

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Definitions

All results presented here were obtained jointly with Mikhail Tkachenko.

All the topologies considered here are assumed to be T_1-topologies on infinite sets.
If $\tau, \sigma \subseteq \mathcal{P}(X)$ are topologies on X, then

$$\tau \wedge \sigma = \tau \cap \sigma,$$

$$\tau \vee \sigma = \text{ the topology generated by } \tau \cup \sigma,$$

$$\mathbf{1}_X = \mathcal{P}(X) \text{ is the discrete topology },$$

$$\mathbf{0}_X = \{X \setminus F : F \in [X]^{<\omega}\} \cup \{\emptyset\} \text{ is the co-finite topology.}$$

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General properties of $\Lambda(\kappa)$

If $\kappa \geq \omega$, then $\Lambda(\kappa)$ denotes the lattice of all T_1-topologies on κ.

(1) (G. Birkhoff, FM 1936) If $\kappa \geq \omega$, then the lattice $\Lambda(\kappa)$ is complete,

(2) (R. W. Bagley J. London Math. Soc. 1955) The lattice $\Lambda(\kappa)$ is not complementary whenever $\kappa \geq \omega$,

(3) (R. W. Bagley and D. Ellis, Math. Japon, 1954) The lattice $\Lambda(\kappa)$ is not distributive whenever $\kappa \geq \omega$.

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Example: let $\kappa \geq \omega$ and let $x_0 \in \kappa$ be fixed. If $A \subseteq \kappa \setminus \{x_0\}$ is infinite and $\kappa \setminus A$ is also infinite, then topologies:

$$\tau_0 = \mathbf{0}_\kappa$$

$$\tau_1 = \mathbf{0}_\kappa \vee \{\emptyset, A, \kappa\}$$

$$\tau_2 = \mathbf{0}_\kappa \vee \{\emptyset, A \cup \{x_0\}, \kappa\}$$

$$\tau_3 = \mathbf{0}_\kappa \vee \{\emptyset, \{x_0\}, \kappa\}$$

$$\tau_4 = \mathbf{0}_\kappa \vee \{\emptyset, A, \{x_0\}, A \cup \{x_0\}, \kappa\}$$

form the lattice N_5 (pentagon), and so the lattice $\Lambda(\kappa)$ is not distributive.

Further definitions

(1) Topologies $\tau, \sigma \subseteq \mathcal{P}(X)$ are called to be transversal if

$$\tau \vee \sigma = \mathbf{1}_X,$$

whereas they are called independent whenever

$$\tau \wedge \sigma = \mathbf{0}_X.$$

(2) Topologies $\tau, \sigma \subseteq \mathcal{P}(X)$ are called complementary if

$$\tau \vee \sigma = \mathbf{1}_X \text{ and } \tau \wedge \sigma = \mathbf{0}_X.$$

(3) Topological spaces (X, τ) and (Y, σ) are called transversal (independent) if there exists a bijection $f\colon X \to Y$ such that τ and

$$f^{-1}(\sigma) = \{f^{-1}[U] \colon U \in \sigma\}$$

are transversal (independent). Moreover, if $(X, \tau) = (Y, \sigma)$, then (X, τ) is called to be self-transversal (self independent).

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Easy observations

Fact 1

Assume (X, τ) is a topological space. If there exists a closed discrete subspace $D \subseteq X$ such that $|D| = |X|$, then (X, τ) is self-transversal.

Indeed, we can assume that $f : X \to X$ is a bijection such that $f[X \setminus D] = D$. Then for every $x \in X \setminus D$ there exist open sets $U, V \subseteq X$ such hat $x \in U$, $U \cap D = \emptyset$ and $f(x) \in V$ and $V \cap D = \{f(x)\}$. Therefore, if

$$\sigma = f^{-1}(\tau) = \{f^{-1}[U] : U \in \tau\},$$

then $\{x\} = U \cap f^{-1}[V] \in \tau \vee \sigma$.

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Fact 2

For every Hausdorff space $\tau \subseteq \mathcal{P}(X)$ there exists a compact Hausdorff topology $\sigma \subseteq \mathcal{P}(X)$ which is transversal to τ.

Indeed, if x, y are distinct points of X, then here exist disjoint infinite sets $A, B \subseteq X$ such that $X = A \cup B$ and $x \in \mathrm{Int}_\tau A$ and $y \in \mathrm{Int}_\tau B$. Then the topology

$$\sigma = \{\{z\} : z \in X \setminus \{x, y\}\} \cup \{\{x\} \cup (B \setminus F) : F \in [X]^{<\omega}\} \\ \cup \{\{y\} \cup (A \setminus F) : F \in [X]^{<\omega}\}$$

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Transversal topologies

Theorem 1

Let X be a countable infinite regular space. Then X is transversal to the space $\mathbb{Q}$ of all rationals iff X is not homeomorphic to the convergent sequence $\{0\} \cup \{\frac{1}{n+1} : n \in \omega\}$.

Lemma

Every non-compact regular countable (infinite) space has a continuous bijection onto the space $\mathbb{Q}$.

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Theorem 2

If the topologies $\tau, \sigma \subseteq \mathcal{P}(X)$ are transversal, then

$$|X| = \max\{\mathrm{nw}(X,\tau), \mathrm{nw}(X,\sigma)\},$$

where nw *denotes the net-weight of a space. Moreover, if the spaces are compact, then*

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Corollary 1

If on X there is a pair of compact metric transversal topologies, then X is countable.

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Independent topologies

Theorem 3

If compact topologies $\tau, \sigma \subseteq \mathcal{P}(X)$ are independent, then for every $A \subseteq X$ we have

$$\textit{either } |\operatorname{cl}_\tau A| \geq 2^{\aleph_1} \textit{ or } |\operatorname{cl}_\sigma A| \geq 2^{\aleph_1}.$$

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No set of power less than $2^{\aleph_1}$ admits a pair of compact independent topologies.

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Theorem 4

Assume $\tau, \sigma \subseteq \mathcal{P}(X)$ are independent topologies and every infinite subset of X contains an infinite subsets with at most finitely many accumulation points in the topology τ. Then every infinite set has infinitely many accumulations points in topology σ. In particular (X, σ) is countably compact.

Corollary 3 (Shakhmatow-Tkachenko-Wilson)

No countable set admits a pair of independent Hausdorff topologies.

Theorem 4

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Complementary topologies

Theorem 5 (S.Watson)

On a set of power $(2^\omega)^+$ there exists a pair of complementary Tychonoff topologies.

Theorem 6 (Shakhmatov-Tkachenko)

On a set of power 2^{2^ω} there exists a pair of complementary compact topologies.

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Theorem 7

Let (X,τ) be Hausdorff space with $|X| = \kappa = \kappa^\omega$ and let

$$\mathrm{Iso}(X,\tau) = \{x \in X : \{x\} \in \tau\}$$

be dense in (X,τ). If every closed (in sense of τ) subset of X is of cardinality κ, then there exists a family $\mathcal{T}$ of locally compact topologies on X such that:

(1) *every $\sigma \in \mathcal{T}$ is complementary to τ,*

(2) *every two distinct elements of $\mathcal{T}$ are transversal,*

(3) $|\mathcal{T}| = |\mathrm{Iso}(X,\tau)|$

Moreover, there exists at least one compact topology complementary to τ.

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Corollary 4

There exist infinitely many locally compact and at least one compact topology complementary to $\beta\mathbb{N}$. In particular on the set of cardinality $2^{2^{\omega}}$ there exists a pair of complementary compact topologies.

Corollary 5

There is consistent with ZFC that on the set of cardinality $2^{\aleph_0}$ there exists a pair of complementary compact topologies.

Indeed, under the assumption that $2^{\aleph_0} = 2^{\aleph_1}$ and $\mathfrak{s} = \aleph_1$, a modification of the Fedorchuk example (asserting that there exists a compact Hausdorff space of cardinality $2^{\aleph_0}$ without nontrivial converging sequences) gives a space satisfying the assumption of the Theorem 7 with $\kappa = 2^{\aleph_0}$.

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Questions:

1. Does there exists (in ZFC) a pair of complementary compact topologies on a set of cardinality $2^{\aleph_1}$?

2. Let $\kappa > 2^{2^\omega}$. Does there exists on a set of cardinality κ a pair of complementary compact topologies?

3. Is it consistent with ZFC, that $(2^\omega)^+ < 2^{2^\omega}$ and there exists a pair of complementary compact topologies on a set of cardinality $(2^\omega)^+$?

4. Is it true that whenever compact topologies $\tau, \sigma \subseteq \mathcal{P}(X)$ are complementary then every point of X is isolated either in τ or in σ?

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Alexandroff duplicate

Definition:
If (X, τ) is a topological space then the Alexandroff duplicate of X consists of the set $A(X) = X \cup X^*$ endowed with the topology τ^*, where $X \cap X^* = \emptyset$ and $\varphi\colon X \to X^*$ is a bijection and

$$\tau^* = \{A \cup [(U \cup U^*) \setminus B] \colon U \in \tau,\ A, B \subseteq X^*,\ |B| < \aleph_0\},$$

where $U^* = \varphi(U)$ for every $U \subseteq X$.

Fact 3

For every space X, the Alexandroff duplicate $A(X)$ is self-transversal.

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Theorem 8

If X is a compact separable space, and $G(X)$ is the Gleason space (=absolute) of X then $A(G(X))$, the Alexandroff duplicate of $G(X)$ has a compact complementary topology.

Theorem 9

If X is a T_1-topological space and $|X| \leq 2^\omega$ then the space

$$X \oplus A(\beta\mathbb{N} \setminus \mathbb{N})$$

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