Three applications of ideal dichotomy

- No S-spaces (assuming Ideal dichotomy for ω_1 generated ideals).
- No Souslin trees (under PID).
- $\mathfrak{b} \leq \omega_2$ (under PID).

Theorem

PFA implies that there no S-spaces. In fact, the simple dichotomy for $\aleph_1$-generated ideals implies that there are no S-spaces.

Recall this dichotomy. If I is any ω_1 generated ideal of countable sets then either there is an uncountable set our of I or an uncountable set inside of I.

Proof. Recall the definition: An S-space is a regular, hereditarily separable, but not hereditarily Lindelof topological space.

To prove that no such space exists (under the dichotomy), suppose that X is a regular topological space which is not hereditarily Lindelof and we shall prove that X is not hereditarily separable. Since X is not hereditarily Lindelof, X has a subspace $S = \{x_\alpha \mid \alpha < \omega_1\}$ such that every initial part $S_\delta = \{x_\alpha \mid \alpha \leq \delta\}$ is open in S (i.e. S is “right-separated”). We consider the subspace topology on S and shall find a subset of S which is not separable.

Since S is regular, each x_α has an open neighborhood U_α with closure $\overline{U}_\alpha \subset S_\alpha$.

These countable closed sets generate an ideal I. By the dichotomy, there is an uncountable set $D \subset S$ which is either “inside” or “out” of I.

If D is in, then every countable subset E of D is in I, which means that it is covered by a countable closed set, and hence E is not dense in D.

If D is out of I, then D has a finite intersection with every set in I. So in particular the intersection of D with every U_α is finite. As S is a Hausdorff space, D is discrete (and therefore not separable). □

Now we deal with Souslin trees under PID.

No Souslin trees under PID

Let T be an ω_1 tree. Define an ideal I by: $A \in I$ iff $A \subset T$ is countable and for every $t \in T$ $A \cap \{x \in T \mid x < t\}$ is finite.

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There are two possibility of the dichotomy:

1. There is an uncountable set in I: this yields an uncountable antichain.
2. There is an uncountable set out of I: this yields an uncountable chain.

Size of continuum under PID

By Todorcevic and Velickovic, the PFA implies the continuum is $\aleph_2$. Now the PID is consistent with CH. So PID does not imply $\mathfrak{c} = \aleph_2$. Does it imply $\mathfrak{c} \leq \aleph_2$? This is an open question. Todorcevic has proved however that the PID implies $\mathfrak{b} \leq \aleph_2$.

Recall that $\mathfrak{b}$ is the smallest cardinality of an unbounded subset of ω^ω in the $<^*$ ordering.

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Definition

Suppose $f <^* g$. Define $\chi(f, g) = n$ if n is the minimal integer such that for all $m \geq n$ $f(m) < g(m)$.

Definition

For $f \in \omega^\omega$ define $(< f) = \{e \in \omega^\omega \mid e <^* f\}$.

Definition

If $A \subset (< g)$, $\lim_{f \in A} \chi(f, g) = \infty$ means that for every n, $\{f \in A \mid \chi(f, g) < n\}$ is finite.

Definition

Let I_g be defined as the collection of all countable $A \subset (< g)$ such that $\lim_{f \in A} \chi(f, g) = \infty$.

Theorem

I_g is a P-ideal

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$g_1 <^* g_2$ implies $I_{g_1} \supseteq I_{g_2}$.

Proof of $\mathfrak{b} \leq \omega_2$

Assume $\mathfrak{b} > \omega_2$. Construct a $<^*$ increasing sequence in ω^ω of length ω_2: $\langle f_\xi \mid \xi < \omega_2 \rangle$.
We define an ideal I of countable subsets of ω_2.

Definition

$X \in I$ iff X is countable and for some $\xi_0 < \omega_2$, for all $\xi_0 \leq \alpha < \omega_2$ we have $X \in I_{f_\xi}$. (By this we mean $\{f_\alpha \mid \alpha \in X\}$.

Lemma

Assuming $\omega_2 < \mathfrak{b}$, I is a P-ideal.

Proof. Suppose $A_i \in I$, for $i \in \omega$. For every $\xi < \omega_2$ high enough every A_i is in I_{f_ξ}. Define $h_\xi(j) = \{\alpha \in A_j \mid \chi(f_\alpha, f_\xi) \leq j\}$ (a finite set).
Then find a single h that dominates all $h_x i$ (by $\omega_2 < \mathfrak{b}$), and use it to define $A = \bigcup_i (A_i \setminus h(i))$.

A contradiction

By the PID there are two possibilities.

(1) There is an uncountable X inside I.

Suppose X has order type ω_1. There is $\xi < \omega_2$ high enough so that for every $X_0 \subset X$ an initial segment $\lim_{\alpha \in X_0} \chi(f_\alpha, f_\xi) = \infty$. But this is impossible as we can fix $\chi(f_\alpha, f_\xi)$ on some uncountable set of $\alpha \in X$.

Second PID possibility: ω_2 is a countable union of sets out of I. So some unbounded $E \subset \omega_2$ is out of I.

Define $g \in \omega^\omega$ so that $f_\xi <^* g$ for all ξ.

Define $s \in (\omega \cup \{\infty\})^\omega$ by

$s(n) = \sup_{\alpha \in E} f_\alpha(n)$.

Claim: s hits ∞ only a finite number of times.

(Otherwise we would find an infinite subset of E in I).

Define $s^-(n) = s(n) - 1$. Then $f_\xi <^* s^-$ for all ξ. Yet we can find now an infinite subset of E that is in I_{s^-} and hence in each I_{f_ξ} and so in I.