# Basis problem for analytic multiple gaps

Antonio Avilés (joint work with S. Todorcevic)

Universidad de Murcia, Author supported by MEyC and FEDER under project MTM2011- 25377

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# Introduction

1. A. Avilés, S. Todorcevic, *Multiple gaps*, Fundamenta Mathematicae 213 (2011), 15-42.

2. A. Avilés, S. Todorcevic, *Finite basis for analytic strong n-gaps*, Combinatorica 33(4) 2013, 375-393

3. A. Avilés, S. Todorcevic, *Basis problem for analytic multiple gaps*, arxiv.org, 100 p.

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1. S. Todorcevic, *Analytic gaps*, Fundamenta Mathematicae 150 (1996), 55-66.

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We are going to look at different classes of subsequences of this sequence.

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- The classes $\mathbf{\Gamma_Q}$ and $\mathbf{\Gamma}^+$ cannot be separated.

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2. Here, disjoint is equivalent to orthogonal: $A \cap B$ is finite whenever $A \in \mathbf{\Gamma_i}$, $B \in \mathbf{\Gamma_j}$ for $i \neq j$.
3. The families $\mathbf{\Gamma_i}$ live in $\mathscr{P}(N) = 2^N$, so they might be Borel, analytic, coanalytic, projective, etc.

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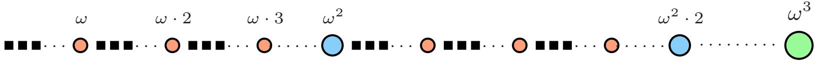
$$\boxed{\forall a_0 \in \mathbf{\Gamma_0}, \ldots, a_{n-1} \in \mathbf{\Gamma_{n-1}} \quad \exists p \quad |a_i \cap N_i^p| < \aleph_0.}$$

# An example of a 3-gap

Consider $N$ the set of successor ordinals below $\omega^3$



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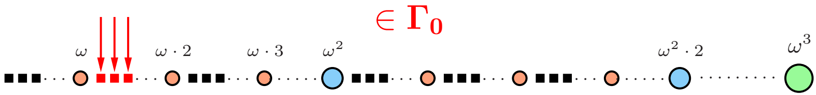


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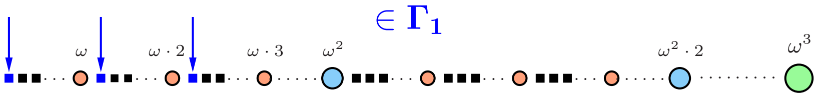


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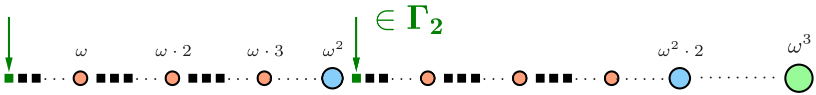


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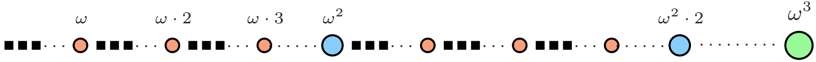


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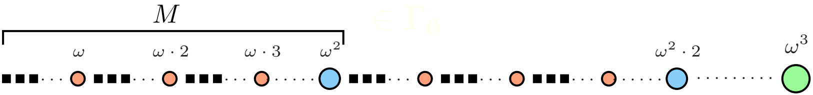


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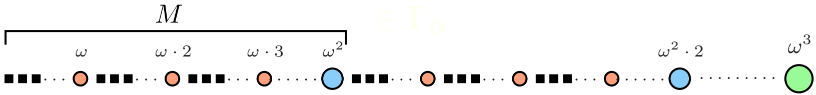


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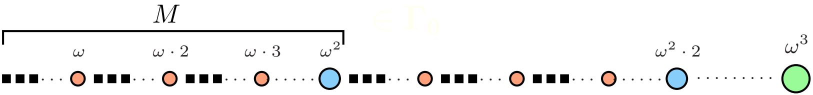


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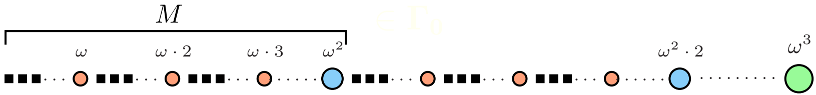


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Can we always isolate a part of a Borel gap from the rest? Some parts, but not all...

# Sample results

**Theorem**

If $\boldsymbol{\Gamma}_{\mathbf{0}}, \ldots, \boldsymbol{\Gamma}_{\mathbf{n-1}}$ is an analytic $n$-gap, then $\exists M \subset N$ and $i < j < n$ :

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$f(3) = 6$, $f(k) = k^2 - k$

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- $\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M, \mathbf{\Gamma_2}|_M$ form a 3-gap.
- $\mathbf{\Gamma_k}|_M = \emptyset$ for all but at most 58 many of the remaining $k$.

$f(3) = 58$, $f(k) \sim \frac{3 \cdot 9^k}{8\sqrt{2\pi k}}$

**Theorem**

If $\mathbf{\Gamma_0}, \ldots, \mathbf{\Gamma_{n-1}}$ is a strong analytic $n$-gap, then $\exists M \subset N$ :

- $\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M, \mathbf{\Gamma_2}|_M$ form a strong 3-gap.
- $\mathbf{\Gamma_k}|_M = \emptyset$ for all but at most 6 many of the remaining $k$.

$f(3) = 6$, $f(k) = k^2 - k$

# Part II
## The first-move structure of the $n$-adic tree and strong gaps

# The $n$-adic tree

The $n$-adic tree is the set $n^{<\omega}$ of finite sequences of $0, 1, \ldots, n-1$

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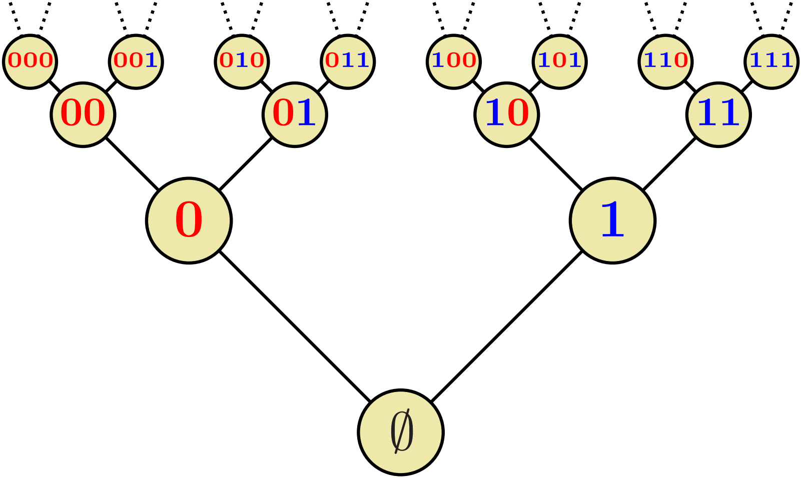


The 2-adic tree

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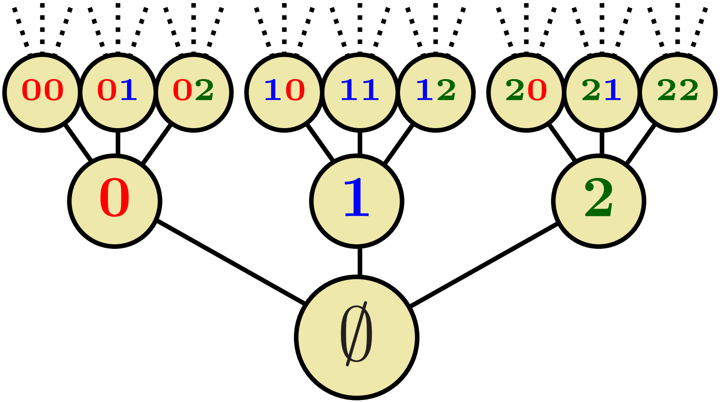


The 3-adic tree

# The first-move structure of $n^{<\omega}$

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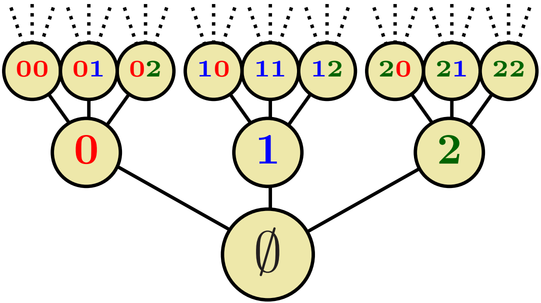


$$\emptyset \prec 0 \prec 1 \prec 2 \prec 00 \prec 01 \prec 02 \prec 10 \prec 11 \prec 12 \prec \cdots$$

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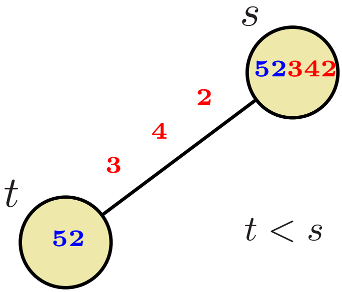

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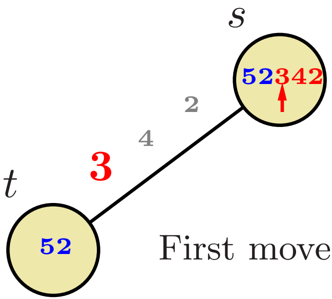


$$\text{First move}(t, s) = 3$$

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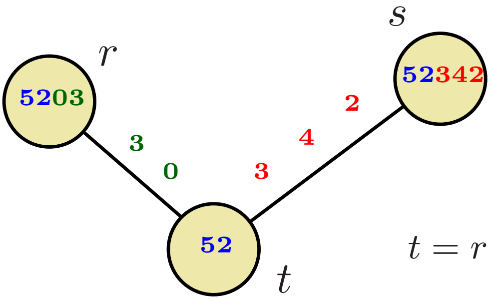


$$t = r \wedge s$$

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**Definition**

The meet-closure $\langle\langle A \rangle\rangle$ of a set $A \subset n^{<\omega}$ is the smallest set which contains $A$ and is closed under the meet operation.

# First-move equivalence

**Let $A$, $B$ be subsets of $n^{<\omega}$.**

A first-move isomorphism between $A$ and $B$ is a bijection $f : A \longrightarrow B$ which extends to a bijection $f : \langle\langle A\rangle\rangle \longrightarrow \langle\langle B\rangle\rangle$ which preserves all relevant characteristics of the first move structure.

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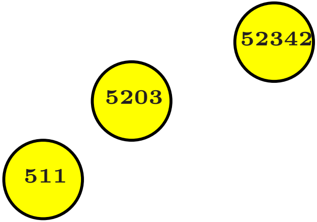

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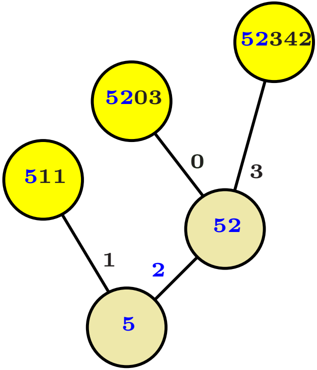

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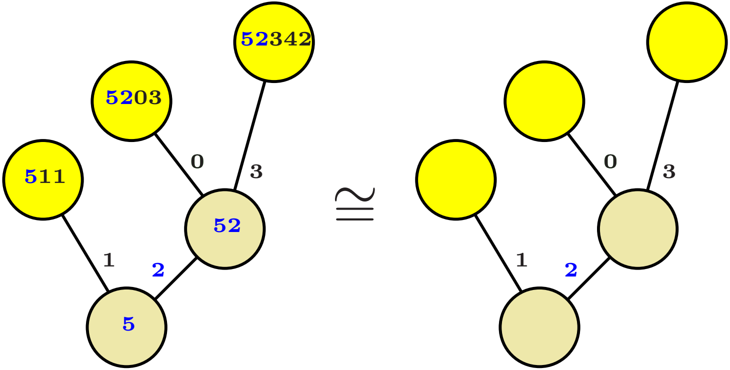

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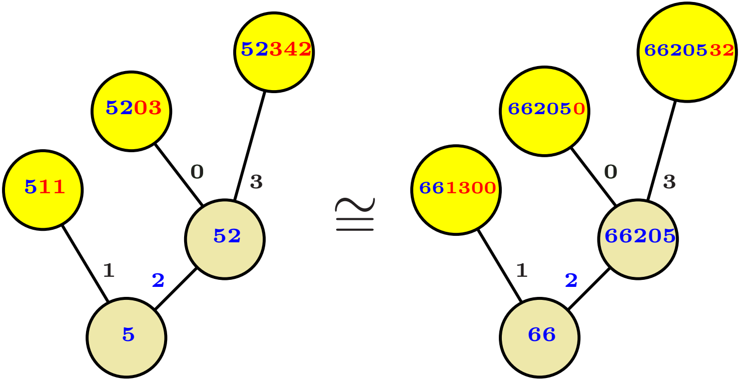

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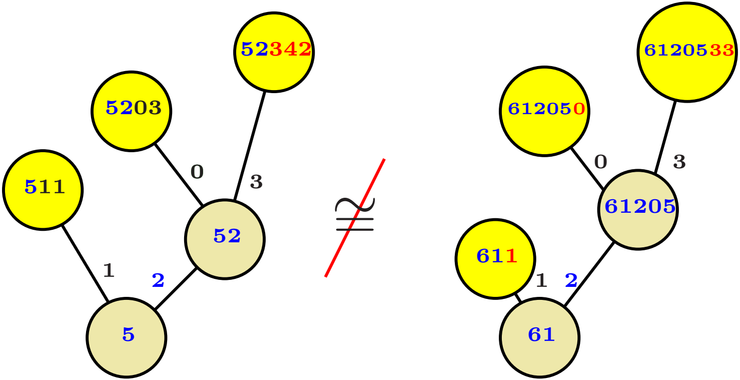

**Theorem**

Fix a set $A \subset n^{<\omega}$, and let $\mathscr{A}$ be the family of all subsets of $n^{<\omega}$ first-move isomorphic to $A$.

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This essentially follows from Milliken's partition theorem.

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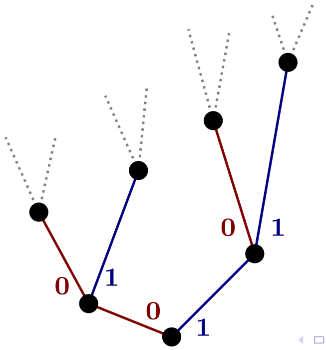

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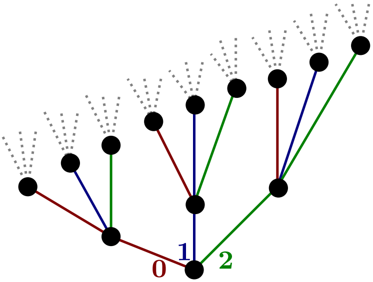

# Combs

For $i, k < n$, an $(i, k)$-comb is a set that is first-move isomorphic to

$$\{(k), (iik), (iiiik), (i^6 k), (i^8 k), (i^{10} k), \ldots\}$$

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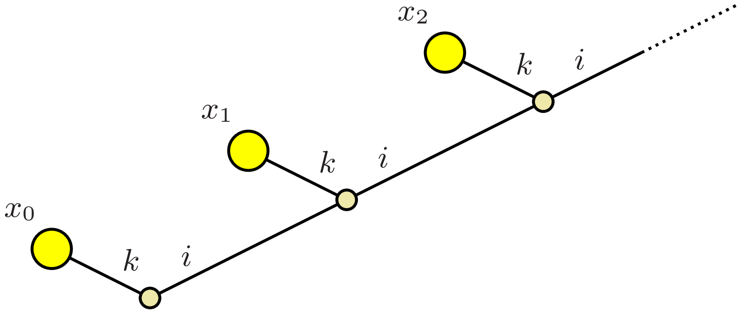
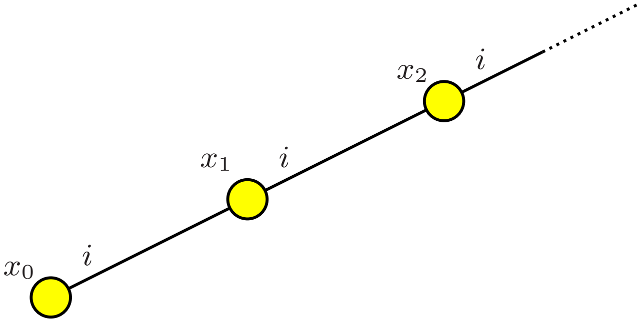


# Combs

An $(i,i)$-comb is called an $i$-chain.



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   We call this a standard strong $n$-gap.

# Finding a standard gap inside

**Theorem**

Let $\{\mathbf{\Gamma_i} : i < n\}$ be a strong analytic gap on $\mathbb{N}$. Then there exists a one-to-one map $u : n^{<\omega} \longrightarrow \mathbb{N}$ such that

- If $A$ is an $i$-chain, then $u(A) \in \mathbf{\Gamma_i}$.

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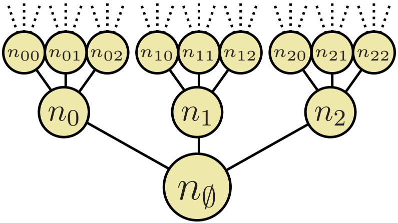


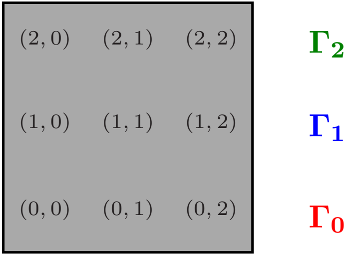

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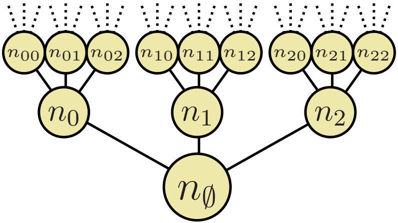


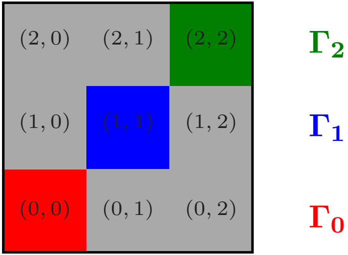

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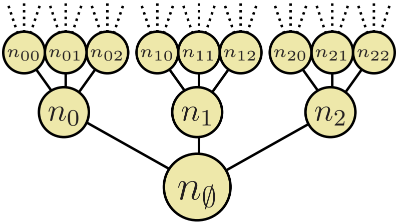


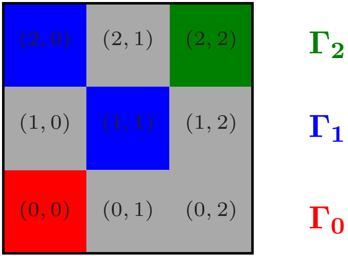

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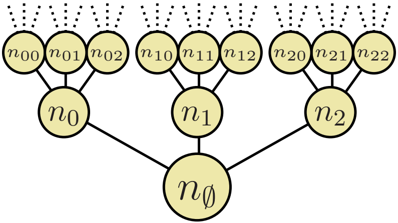


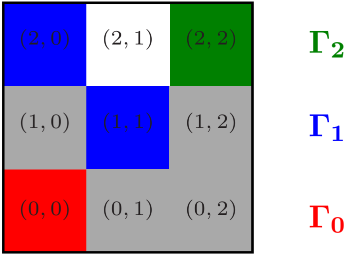

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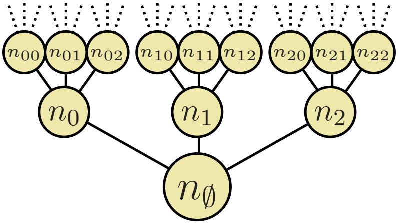


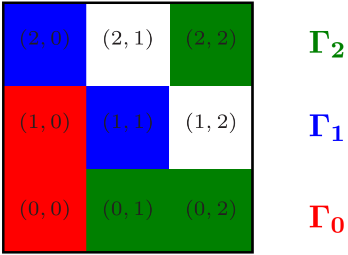

# Finite basis for strong $n$-gaps

**Theorem**

Let $\{\boldsymbol{\Gamma}_{\mathbf{i}} : i < n\}$ be a strong analytic gap on $\mathbb{N}$. Then there exists

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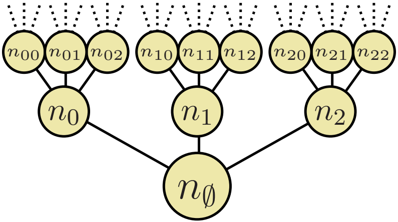
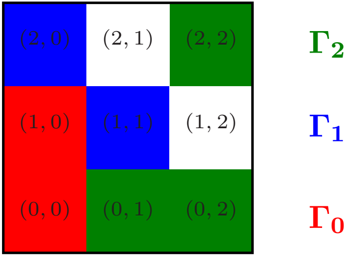
2. a standard strong $n$-gap $\{\boldsymbol{\Delta}_{\mathbf{i}} : i < n\}$

such that $u(A)$ contains an infinite set from $\boldsymbol{\Gamma}_{\mathbf{i}}$ if and only if $A$ contains an infinite set from $\boldsymbol{\Delta}_{\mathbf{i}}$



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- Inside the standard strong gaps, there are the minimal ones
  - $\Delta$ is minimal if $\mathbf{E} \leq \mathbf{\Delta} \Rightarrow \mathbf{\Delta} \leq \mathbf{E}$.
  - Two minimal are equivalent if $\mathbf{\Delta}' \leq \mathbf{\Delta}$ and $\mathbf{\Delta} \leq \mathbf{\Delta}'$

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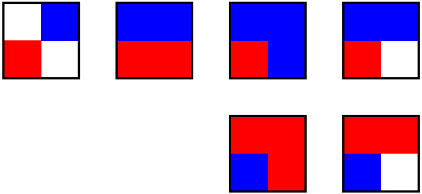
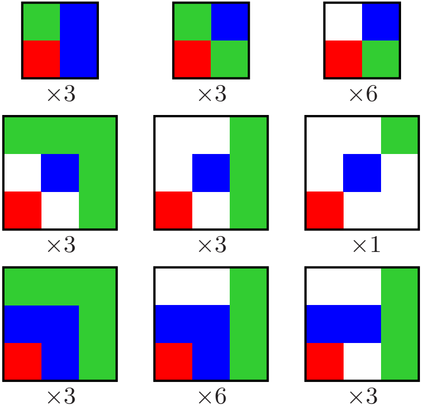
- The category formed by sets $n \times n$ and morphisms as above governs the behavior of strong analytic $n$-gaps.
- This allows to compute the minimal strong $n$-gaps: each of them is given by seven parameters $(A, B, C, D, E, \psi, \gamma)$

# Minimal strong gaps



Minimal analytic strong 2-gaps

# Minimal strong gaps




Minimal analytic strong 3-gaps

# Part III

# The record structure of the $n$-adice tree and general gaps

# The set of records from $t$ to $s$

Let $t < s$ be in $n^{<\omega}$,

## The set of records from $t$ to $s$

Let $t < s$ be in $n^{<\omega}$, $s = (t_0, \ldots, t_n, r_0, \ldots, r_m)$

# The set of records from $t$ to $s$

Let $t < s$ be in $n^{<\omega}$, $s = (t_0, \dots, t_n, r_0, \dots, r_m)$

**Definition**

A record node from $t$ to $s$ is a node $(t_0, \dots, t_n, r_0, \dots, r_{k-1})$ such that $r_k > r_i$ for all $i < k$.

# The set of records from $t$ to $s$

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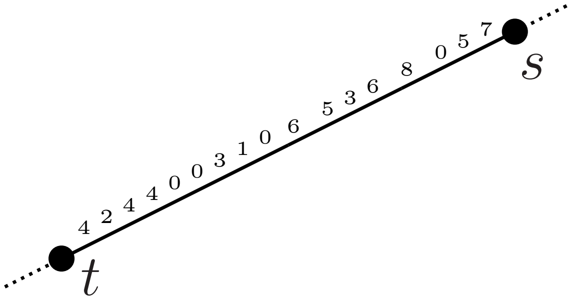

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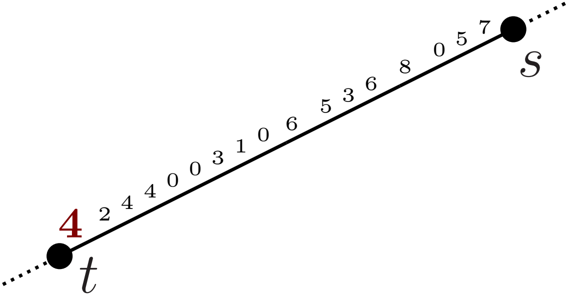

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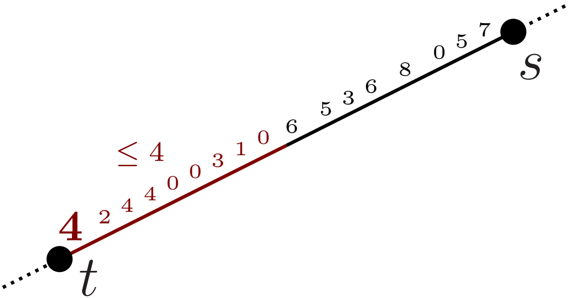

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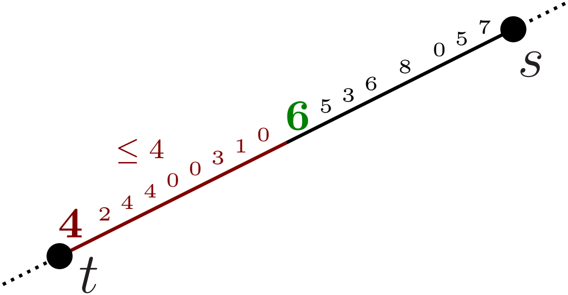

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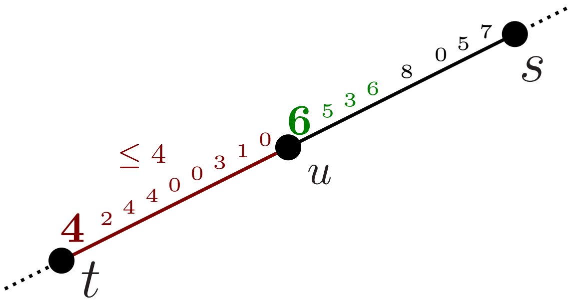

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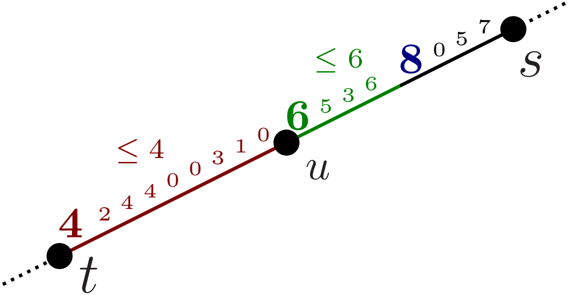

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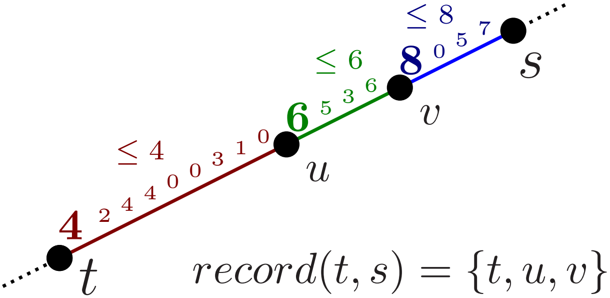

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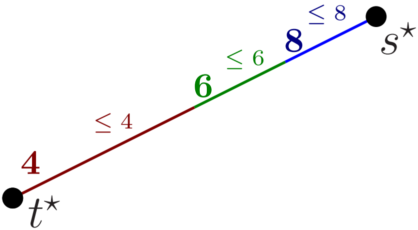
- A record isomorphism between $A$ and $B$ is a bijection $f : A \longrightarrow B$ which extends to a bijection $f : \langle A \rangle \longrightarrow \langle B \rangle$ which preserves all relevant characteristics of the record structure.

# Record equivalence

A set $\{t^\star, s^\star\}$ record-isomorphic to $\{t, s\}$ as before:



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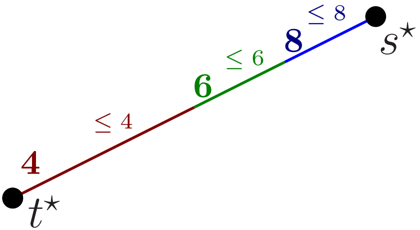


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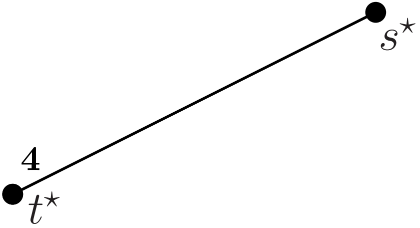

# Ramsey theorem

**Theorem**

Fix a set $A \subset n^{<\omega}$, and let $\mathscr{A}$ be the family of all subsets of $n^{<\omega}$ record isomorphic to $A$. If $c : \mathscr{A} \longrightarrow m$ is measurable, then there exists $T \subset n^{<\omega}$ such that

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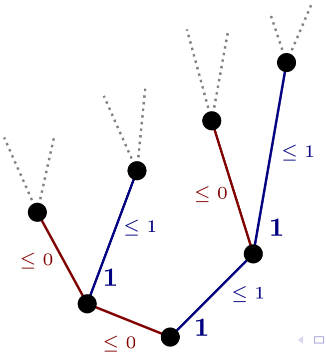

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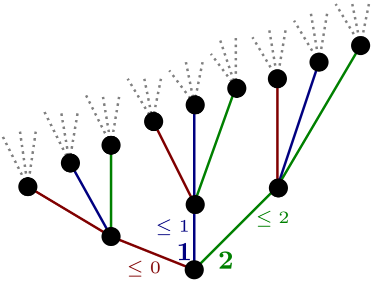

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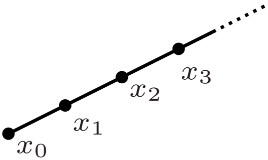
2. Comb-types are given by two increasing sequences of numbers $< n$, that we write in two rows, with a global order, which is read from left to right, like $[{}_3{}^{03}{}_5]$, $[{}^1{}_4{}^3{}_{67}]$, etc

   (the rightmost number must always be in the lower row, and the leftmost numbers of each row must be different)

# A set $\{x_0, x_1, x_2, \ldots\}$ of type [468]



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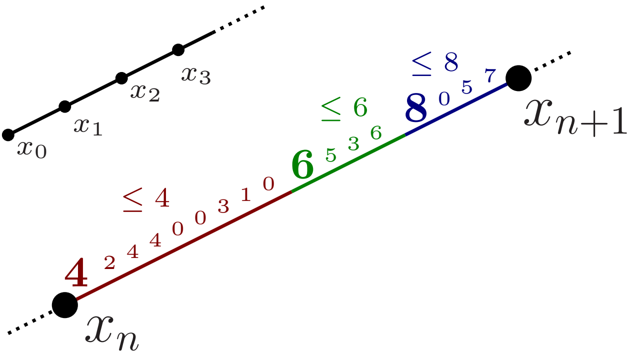

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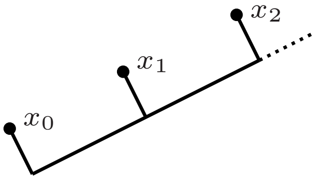

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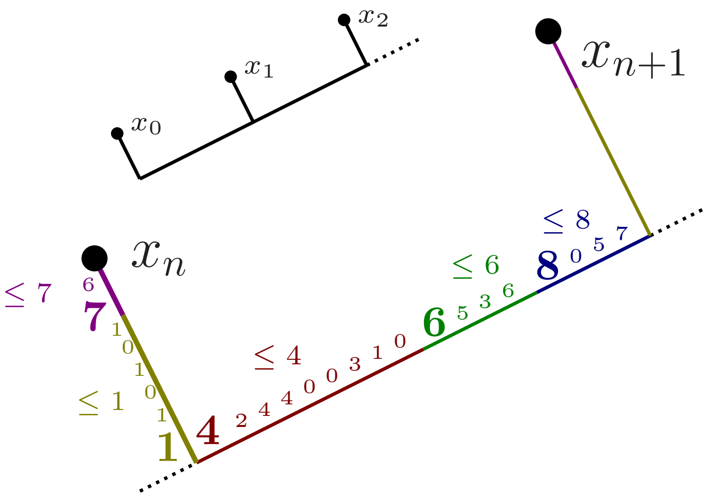

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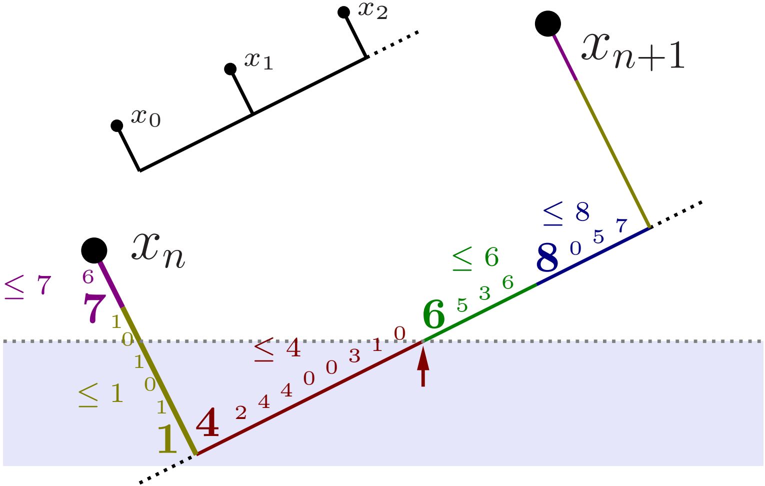

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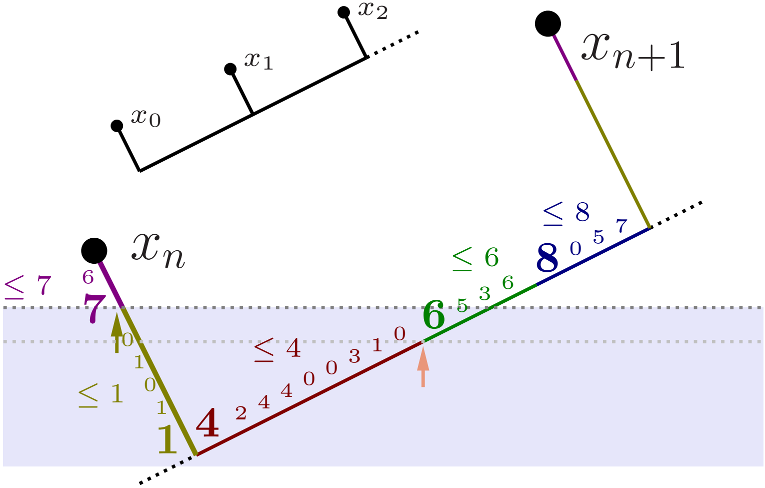

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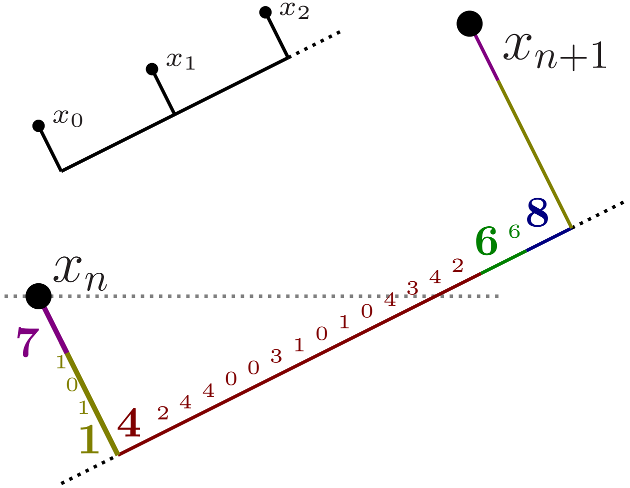

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There are eight types in $2^{<\omega}$:

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There are approximately $\sim \frac{3 \cdot 9^n}{8\sqrt{2\pi n}}$ types in $n^{<\omega}$

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   We call this a standard $n$-gap.

# Finding a standard gap inside

**Theorem**

Let $\{\Gamma_i : i < n\}$ be an analytic gap on $\mathbb{N}$. Then there exists a one-to-one map $u : n^{<\omega} \longrightarrow \mathbb{N}$ and a permutation $\varepsilon$ such that

- If $A$ is an $[i]$-chain, then $u(A) \in \Gamma_{\varepsilon(i)}$.

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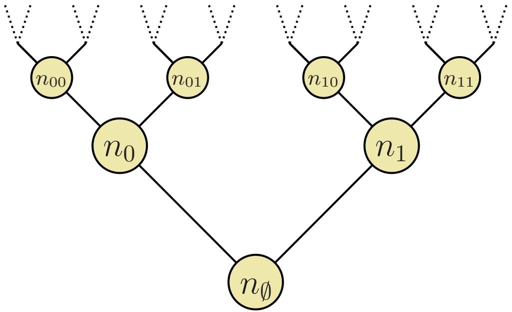


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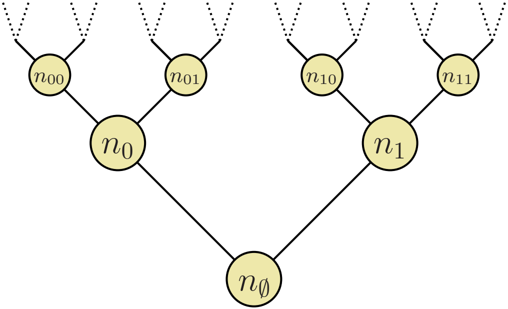


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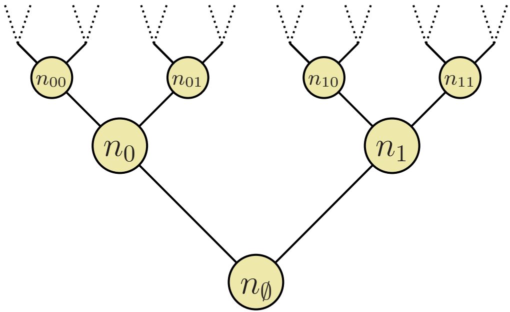


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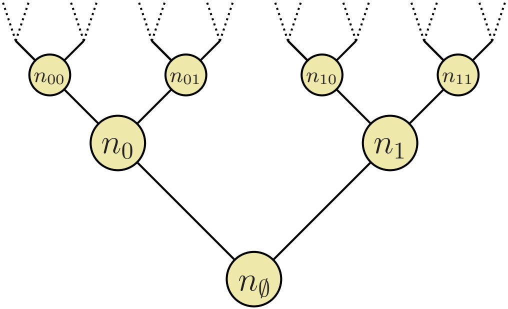


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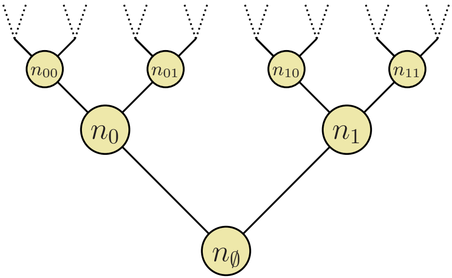


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- Inside the standard $n$-gaps, there are the minimal ones
  - $\Delta$ is minimal if $\mathbf{E} \leq \mathbf{\Delta} \Rightarrow \mathbf{\Delta} \leq \mathbf{E}$.
  - Two minimal are equivalent if $\mathbf{\Delta}' \leq \mathbf{\Delta}$ and $\mathbf{\Delta} \leq \mathbf{\Delta}'$

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Problems about general analytic gaps are reduced to problems about standard gaps, which in turn reduce to finite combinatorial problems.

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Let $\mathfrak{T}_n$ be the set of types in $n^{<\omega}$.

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A function $f : \mathfrak{T}_n \longrightarrow \mathfrak{T}_m$ is a morphism if there exists a one-to-one $u : n^{<\omega} \longrightarrow m^{<\omega}$ which sends sets of type $\tau$ to sets of type $f\tau$.

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- This category is more complex than the one for strong gaps, so we were not able to describe the minimal analytic $n$-gaps.
- We studied some phenomena in this category,

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A function $f : \mathfrak{T}_n \longrightarrow \mathfrak{T}_m$ is a morphism if there exists a one-to-one $u : n^{<\omega} \longrightarrow m^{<\omega}$ which sends sets of type $\tau$ to sets of type $f\tau$.

- The category formed by the sets $\mathfrak{T}_n$ and morphisms as above governs the behavior of analytic $n$-gaps.
- This category is more complex than the one for strong gaps, so we were not able to describe the minimal analytic $n$-gaps.
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# Finite combinatorics behind

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## The minimal analytic 2-gaps

There are 9 minimal 2-gaps (5 up to permutation):

| | $\Gamma_0$ | $\Gamma_1$ |
|---|---|---|
| $1^{**}$ | $[0]$ | all other types |
| $2^{**}$ | $[0]$ | $[1]$ |
| $3^{**}$ | $[0]$ | $[1], [01]$ |
| $4^{*}$ | $[0], [01]$ | $[1]$ |
| $5^{**}$ | $[0]$ | $[1], [01], [^{1}{}_{01}]$ |

$^{**}$: two permutations

$*$: equivalent to its permutation

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1. $\max(\tau_0) \leq \max(\tau_1) \leq \cdots \leq \max(\tau_{n-1})$,
2. There exists a morphism $f : \mathfrak{T}_n \longrightarrow \mathfrak{T}_m$ such that $f[i] = \tau_i$.

# Proof of the max function theorem



fact provides a nice embedding $u$ such that $\phi u$ satisfies condition (2) of normal embeddings for all 4-families. This finishes the proof by Lemma 1.1. $\square$

## 2. The max function

Given a type $\tau$, $\max(\tau)$ denotes the maximal number which appears in $\tau$. That is,

$$\max(\tau) = \max(\max(\tau^0), \max(\tau^1)).$$

THEOREM 2.1. *For a family $\{\tau_i : i \in n\} \subset \mathfrak{T}_m$ the following are equivalent:*

(1) *There exists a normal embedding $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ such that $\bar{\phi}[i] = \tau_i$,*
(2) $\max(\tau_0) \leq \cdots \leq \max(\tau_{n-1})$.

PROOF. Suppose that item (1) holds, pick $i < j$ and let us check that $\max(\tau_i) \leq \max(\tau_j)$. Let $\alpha = \phi(j) \wedge \phi(ji)$. Since $\{j, ji\}$ are the two first element of a chain of type $[i]$, it follows that

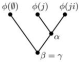
$$(\mathtt{I}) \quad \max(\tau_i) = \max\{\max\{\phi(j) \setminus \alpha\}, \max\{\phi(ji) \setminus \alpha\}\}.$$

On the other hand, both $\{\emptyset, j\}$ and $\{\emptyset, ji\}$ are the beginning of chains of type $[j]$, so if $\beta = \phi(\emptyset) \wedge \phi(j)$ and $\gamma = \phi(\emptyset) \wedge \phi(ji)$ we have similar formulas

$$(\mathtt{II}) \quad \max(\tau_j) = \max\{\max\{\phi(\emptyset) \setminus \beta\}, \max\{\phi(j) \setminus \beta\}\}.$$

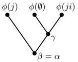
$$(\mathtt{III}) \quad \max(\tau_j) = \max\{\max\{\phi(\emptyset) \setminus \gamma\}, \max\{\phi(ji) \setminus \gamma\}\}.$$

We distinguish three cases. The first case is $\beta < \alpha$, which implies that $\gamma = \beta < \alpha$,




so $\max(\phi(j) \setminus \alpha) \leq \max(\phi(j) \setminus \beta)$ and $\max(\phi(ji) \setminus \alpha) \leq \max(\phi(ji) \setminus \gamma)$ so we conclude from the formulas (I), (II) and (III) above that $\max(\tau_i) \leq \max(\tau_j)$ as desired.

The second case is that $\beta = \alpha$, which implies that $\gamma \geq \alpha = \beta$.




By formula (I), it is enough to check that $\max(\phi(j) \setminus \alpha) \leq \max(\tau_j)$ and $\max(\phi(ji) \setminus \alpha) \leq \max(\tau_j)$. In this case, $\phi(j) \setminus \alpha = \phi(j) \setminus \beta$ so it is clear that $\max(\phi(j) \setminus \alpha) \leq \max(\tau_j)$ by (II). On the other hand,

$$\phi(ji) \setminus \alpha = (\gamma \setminus \alpha)^\frown(\phi(ji) \setminus \gamma).$$

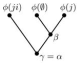
On one side, $\phi(\emptyset) \setminus \beta = (\gamma \setminus \beta)^\frown(\phi(\emptyset) \setminus \gamma)$, therefore

$$\max(\gamma \setminus \alpha) = \max(\gamma \setminus \beta) \leq \max(\phi(\emptyset) \setminus \beta) \leq \max(\tau_j)$$

by (II), and on the other side $\max(\phi(ji) \setminus \gamma) \leq \max(\tau_j)$ by (III), so we conclude that $\max(\phi(ji) \setminus \alpha) \leq \max(\tau_j)$. By formula (I), this finishes the second case.

The third case is that $\beta > \alpha$, which implies that $\gamma = \alpha < \beta$.






This is solved in a similar way as in the second case, changing the role of $j$ and $ji$. By formula (I), it is enough to check that $\max(\phi(j) \setminus \alpha) \leq \max(\tau_j)$ and $\max(\phi(ji) \setminus \alpha) \leq \max(\tau_j)$. Now, $\phi(ji) \setminus \alpha = \phi(ji) \setminus \gamma$ so it is clear that $\max(\phi(ji) \setminus \alpha) \leq \max(\tau_j)$ by (III). On the other hand,

$$\phi(j) \setminus \alpha = (\beta \setminus \alpha)^\frown(\phi(j) \setminus \beta)$$

On one side, $\phi(\emptyset) \setminus \gamma = (\beta \setminus \gamma)^\frown(\phi(\emptyset) \setminus \beta)$ so

$$\max(\beta \setminus \alpha) = \max(\beta \setminus \gamma) \leq \max(\phi(\emptyset) \setminus \gamma) \leq \max(\tau_j)$$

by (III), and on the other side $\max(\phi(j) \setminus \beta) \leq \max(\tau_j)$ by (II). So we conclude that $\max(\phi(j) \setminus \alpha) \leq \max(\tau_j)$ and this finishes the third case.

Now, suppose that (2) holds[1]. For every $i$ fix $(u_i, v_i)$ a rung of type $\tau_i$ and write $u_i = \breve{u}_i{}^\frown\vec{u}_i$ in such a way that $|\breve{u}_i| = |v_i|$. When $\tau_i$ is a chain type, $v_i = \breve{u}_i = \emptyset$ and $\vec{u}_i = u_i$. When $\tau_i$ is a comb type we can make the additional assumption[2] that the last integer of $\breve{u}_i$ and the first integer of $\vec{u}_i$ are both equal to 0. We shall construct an embedding $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ together with auxiliary functions $\phi_i, \phi^i : n^{<\omega} \longrightarrow m^{<\omega}$ for $i = 0, \ldots, n-1$. All of them will be defined by induction on the $\prec$- order of $n^{<\omega}$. We first choose $\phi(\emptyset)$, $\phi_i(\emptyset)$, $\phi^i(\emptyset)$. Let $\{j_1, \ldots, j_p\}$ be an enumeration of all indices $i$ such that $\tau_i$ is a comb type and such that

$$\max(\tau^1_{j_1}) \geq \max(\tau^1_{j_2}) \geq \cdots \geq \max(\tau^1_{j_p}),$$

and moreover, if $\max(\tau^1_{j_r}) = \max(\tau^1_{j_s})$, then $j_r < j_s$ if and only if $r > s$.

We define

$$\begin{array}{rcl}
\phi_{j_1}(\emptyset) & = & \emptyset, \\
\phi_{j_k}(\emptyset) & = & v_{j_1}^\frown \cdots ^\frown v_{j_{k-1}} \\
\phi(\emptyset) & = & v_{j_1}^\frown \cdots ^\frown v_{j_p} \\
\phi^i(\emptyset) & = & \phi_i(\emptyset)^\frown \breve{u}_i{}^\frown 0^{l_i} \text{ if } \tau_i \text{ is a comb type.} \\
\phi_i(\emptyset) & = & \phi^i(\emptyset) = \phi(\emptyset) \text{ if } \tau_i \text{ is a chain type,}
\end{array}$$

The number $l_i$ of 0's added to construct $\phi^i(\emptyset)$ is chosen so that $\phi^i(\emptyset)$ has length strictly larger than $\phi(\emptyset)$. Figure 1 represents how $\phi(\emptyset)$, $\phi_k(\emptyset)$ and $\phi^k(\emptyset)$ look like in the tree. The pattern reflected in this picture will be repeated for $\phi(x)$, $\phi_k(x)$ and $\phi^k(x)$ for any $x$. It is natural to make the notational convention that $\phi_{j_{p+1}} = \phi$ and this will avoid repeating some arguments along the proof.

[1] The proof of later Lemma 5.5 may be enlightening about the necessity of constructing $\phi$ in such a complicated way.

[2] The aim of this assumption is to make sure that the critical nodes of $u_i$ are far away from the splitting between $\breve{u}_i$ and $\vec{u}_i$ and to avoid in this way peculiar situations.

# Proof of the max function theorem



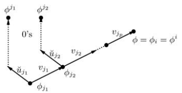


FIGURE 1. Configuration of $\phi$, $\phi_k$, $\phi^k$

We shall see how to define all these functions on $x^\frown k$ once they are defined on all $y \prec x^\frown k$, in particular on $y = x$. We consider

$$q = q(k) = \min\{r : \max(\tau^1_{j_r}) < \max(\tau_k) \text{ or } j_r \le k\}$$

(If there is no $r$ like that we may assign the value $q = p+1$). The definition of the functions is then made as follows:

$$\begin{aligned}
\phi(x^\frown k) &= \phi^k(x)^\frown \vec{u}_k{}^\frown v_{j_q}{}^\frown v_{j_{q+1}}{}^\frown \cdots {}^\frown v_{j_p} \\
\phi_{j_r}(x^\frown k) &= \phi_{j_r}(x) \text{ if } r < q \\
\phi_{j_r}(x^\frown k) &= \phi^k(x)^\frown \vec{u}_k{}^\frown v_{j_q}{}^\frown v_{j_{q+1}}{}^\frown \cdots {}^\frown v_{j_{r-1}} \text{ if } r \ge q \\
\phi^i(x^\frown k) &= \phi_i(x^\frown k)^\frown \breve{u}_i{}^\frown 0^{l'_i} \text{ if } \tau_i \text{ is a comb type,} \\
\phi^i(x^\frown k) &= \phi_i(x^\frown k) = \phi(x^\frown k) \text{ if } \tau_i \text{ is a chain type.}
\end{aligned}$$

Now, the number $l'_i$ of 0's added to construct $\phi^i(x^\frown k)$ is chosen so that $\phi^i(x^\frown k)$ has length larger than $\phi(x^\frown k)$ but also larger than all $\phi(y)$, $\phi_j(y)$, $\phi^j(y)$ that have been already constructed for $y \prec x^\frown k$. A picture of what is going on is given by Figure 2. The point is that both sets $\{\phi(x), \phi_k(x), \phi^k(x), k < \omega\}$ and $\{\phi(x^\frown k), \phi_k(x^\frown k), \phi^k(x^\frown k), k < \omega\}$ must follow the pattern[3] provided by Figure 1, but we make $\phi_{j_r}(x^\frown k)$ to stay the same as $\phi_{j_r}(x)$ for $r < q$, while $\phi_{j_r}(x)$ is moved above $\phi^k(x)^\frown \vec{u}_k$ for $r \ge q$.

Claim 1: For every $x \in n^{<\omega}$ and[4] for $r = 1, \ldots, p$,

($\star$) $\phi_{j_{r+1}}(x) = \phi_{j_r}(x)^\frown v_{j_r}{}^\frown w$ for some $w$ such that $\max(w) \le \max(v_{j_r})$.

Proof of Claim 1: This holds when $x = \emptyset$. We suppose that it holds for $x$ and we prove it for $x^\frown k$. For $r < q = q(k)$ we have that $\phi_{j_r}(x^\frown k) = \phi_{j_r}(x)$ while for $r \ge q$ we have that

$$\phi_{j_r}(x^\frown k) = \phi^k(x)^\frown \vec{u}_k{}^\frown v_{j_q}{}^\frown v_{j_{q+1}}{}^\frown \cdots {}^\frown v_{j_{r-1}}$$

[3]When we say *following the same pattern*, we mean up to equivalence. Looking at Figure 2, one may wonder if the long path from $\phi_{j_{q-1}}(x^\frown k)$ till $\phi_{j_q}(x^\frown k)$ is really equivalent to $v_{j_{q-1}}$ as Figure 1 suggests. This is the content of Claim 1.

[4]Remember our convention that $\phi_{j_{p+1}}(x) = \phi(x)$



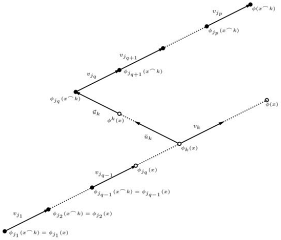


FIGURE 2. Passing from $x$ to $x^\frown k$

Thus, we have $\phi_{j_{r+1}}(x) = \phi_{j_r}(x)^\frown v_{j_r}$ when either $r < q-1$ or $r \ge q$. Only the case when $r = q-1$ deserves special attention. In this case

$$\phi_{j_r}(x^\frown k) = \phi_{j_{q-1}}(x^\frown k) = \phi_{j_{q-1}}(x),$$

$$\phi_{j_{r+1}}(x^\frown k) = \phi_{j_q}(x^\frown k) = \phi^k(x)^\frown \vec{u}_k.$$

Either $\tau_k$ is a chain type (in which case $\phi_k(x) = \phi(x)$) or $k = j_l$ for some $l$ which must satisfy $l \ge q$ by the definition[5] of $q$. In either case the inductive hypothesis implies that [6] $\phi_k(x) = \phi_{j_{q-1}}(x)^\frown v_{j_{q-1}}{}^\frown w_1$ where $\max(w_1) \le \max(v_{j_{q-1}})$. If $\tau_k$ is a chain type, then $\phi^k(x) = \phi_k(x)$, so

$$\phi_{j_q}(x^\frown k) = \phi^k(x)^\frown \vec{u}_k = \phi_{j_{q-1}}(x)^\frown v_{j_{q-1}}{}^\frown w_1{}^\frown \vec{u}_k$$

and this is what we were looking for because[7]

$$\max(\vec{u}_k) \le \max(\tau_k) \le \max(\tau^1_{j_{q-1}}) = \max(v_{j_{q-1}}).$$

On the other hand, if $\tau_k$ is a comb type, then $\phi^k(x) = \phi_k(x)^\frown \breve{u}_k{}^\frown 0^{l'_k}$, so

$$\phi_{j_q}(x^\frown k) = \phi^k(x)^\frown \vec{u}_k = \phi_{j_{q-1}}(x)^\frown v_{j_{q-1}}{}^\frown w_1{}^\frown \breve{u}_k{}^\frown 0^{l'_k}{}^\frown \vec{u}_k$$

and this is again what we were looking for, because[7]

$$\max(\vec{u}_k), \max(\breve{u}_k) \le \max(\tau_k) \le \max(\tau^1_{j_{q-1}}) = \max(v_{j_{q-1}})$$

similarly as in the previous case. This finishes the proof of Claim 1.

[5]If $j_l = k$ then in particular $j_l \le k$ so by the minimality of $q$ in its definition, $q \le l$.

[6]Just apply the formula ($\star$) repeatedly for $r = q-1, q, \ldots$ till arriving at $\phi_k(x)$.

[7]The central inequality $\max(\tau_k) \le \max(\tau^1_{j_{q-1}})$ follows from the definition of $q$

# Proof of the max function theorem



Claim 2: Suppose that $\tau_k$ is a chain type. Then for every $x \in n^{<\omega}$ and every $w \in W_k$, we have that $\phi(x^\frown w) = \phi(x)^\frown u_k{}^\frown w'$ where $\max(w') \leq \max(\tau_k)$.

Proof of Claim 2: We proceed by induction on the length of $w$. Together with the statement of the claim, we shall also prove that for every $i = 0, \dots, k$, we can write $\phi_i(x^\frown w) = \phi(x)^\frown u_k{}^\frown w'_i$ where $\max(w'_i) \leq \max(\tau_k)$. The first case is that $w = (k)$. Remember that

$$\phi(x^\frown k) = \phi^k(x)^\frown \vec{u}_k{}^\frown v_{j_q}{}^\frown v_{j_{q+1}}{}^\frown \cdots {}^\frown v_{j_p}$$

and since $\tau_k$ is a chain type, $\phi^k(x) = \phi(x)$ and $\vec{u}_k = u_k$. Moreover, by the definition of $q = q(k)$ and the way that the sequence $\{j_r\}$ is chosen we have that[8]

$$(\star\star) \quad \max(v_{j_p}) \leq \cdots \leq \max(v_{j_q}) \leq \max(\tau_k)$$

so the expression above is as desired, and the claim is proven for $w = (k)$. Concerning $\phi_i(x^\frown k)$, if $\tau_i$ is a chain type, $\phi_i(x^\frown k) = \phi(x^\frown k)$ and there is nothing to prove. The other case is that $i = j_r$ for some $r$. Then, by the definition of $q$, $r \geq q$ since $j_r = i \leq k$, therefore

$$\phi_i(x^\frown k) = \phi_{j_r}(x^\frown k) = \phi^k(x)^\frown \vec{u}_k{}^\frown v_{j_q}{}^\frown v_{j_{q+1}}{}^\frown \cdots {}^\frown v_{j_{r-1}}$$

In the same way as before, by $(\star\star)$ above, this provides an expression $\phi_i(x^\frown k) = \phi(x)^\frown u_k{}^\frown w'_i$ where $\max(w'_i) \leq \max(\tau_k)$. This finishes the initial step of the inductive proof when $w = (k)$.

Now we assume that our statement holds for $w \in W_k$, we fix $\xi \in \{0, \dots, k\}$ and we shall prove that the statement holds for $w^\frown \xi$. First,

$$(\diamondsuit) \ \phi(x^\frown w^\frown \xi) = \phi^\xi(x^\frown w)^\frown \vec{u}_\xi{}^\frown v_{j_{q(\xi)}}{}^\frown v_{j_{q(\xi)+1}}{}^\frown \cdots {}^\frown v_{j_p}$$

Notice that $\max(\vec{u}_\xi) \leq \max(\tau_\xi) \leq \max(\tau_k)$, and in the same way as we had the expression $(\star\star)$, the defining formula of $q(\xi)$ implies that

$$(\star\star)' \quad \max(v_{j_p}) \leq \cdots \leq \max(v_{j_{q(\xi)}}) \leq \max(\tau_\xi)$$

so all vectors $v_{j_r}$ appearing in the expression $(\diamondsuit)$ above are bounded by $\max(\tau_\xi) \leq \max(\tau_k)$. Hence, the expression $(\diamondsuit)$ above can be rewritten as

$$\phi(x^\frown w^\frown \xi) = \phi^\xi(x^\frown w)^\frown w' \text{ with } \max(w') \leq \max(\tau_k)$$

If $\tau_\xi$ is a chain type, then $\phi^\xi(x^\frown w) = \phi(x^\frown w)$ and we are done, by the inductive hypothesis. If $\tau_\xi$ is a comb type, then

$$\phi^\xi(x^\frown w) = \phi_\xi(x^\frown w)^\frown \breve{u}_\xi{}^\frown 0^{l'_i}$$

which also provides the desired form because $\max(\breve{u}_\xi{}^\frown 0^{l'_i}) \leq \max(\tau_\xi) \leq \max(\tau_k)$ and we can apply the inductive hypothesis to $\phi_\xi(x^\frown w)$.

Finally, we fix $i \in \{0, \dots, k\}$ and we prove that also $\phi_i(x^\frown w^\frown \xi)$ is of the form $\phi(x)^\frown u_k{}^\frown w'_i$ with $\max(w'_i) \leq \max(\tau_k)$. If $\tau_i$ is a chain type, there is nothing to prove because $\phi_i = \phi$. Otherwise $\phi_i$ is a comb type, and $i = j_r$ for some $r$. If $r < q(\xi)$ then $\phi_i(x^\frown w^\frown \xi) = \phi_i(x^\frown w)$ and we apply directly the inductive hypothesis. If $r \geq q(\xi)$, then

$$\phi_i(x^\frown w^\frown \xi) = \phi^\xi(x^\frown w)^\frown \vec{u}_\xi{}^\frown v_{j_{q(\xi)}}{}^\frown v_{j_{q(\xi)+1}}{}^\frown \cdots {}^\frown v_{j_{r-1}}$$

[8] By the definition of $q$, either $\max(v_{j_q}) < \max(\tau_k)$ or $j_q \leq k$. In the latter case, $\max(v_{j_q}) \leq \max(\tau_{j_q}) \leq \max(\tau_k)$ by the statement (2) of Theorem 2.1 that we are assuming.



By the expression $(\star\star)'$ above, all vectors to the right of $\phi^\xi(x^\frown w)$ are bounded by $\max(\tau_\xi) \leq \max(\tau_k)$, while

$$\phi^\xi(x^\frown w) = \phi_\xi(x^\frown w)^\frown \breve{u}_\xi{}^\frown 0^{l'_\xi}$$

is of the form $\phi(x)^\frown u_k{}^\frown w'$ with $\max(w') \leq \max(\tau_k)$, by the inductive hypothesis. This finishes the proof of Claim 2.

Claim 3: Suppose that $\tau_k$ is a comb type, $x \in n^{<\omega}$ and $w \in W_k$. Then

$$\phi_k(x^\frown w) = \phi^k(x)^\frown \vec{u}_k{}^\frown w'$$

where $\max(w') \leq \max(\tau_k^0)$.

Proof of Claim 3: Since $\tau_k$ is a comb type, $k = j_r$ for some $r$. We proceed by induction on the length of $w$. The first case is that $w = (k)$. Notice that $r \geq q = q(k)$ because $j_r = k \leq k$ (by the definition of $q$), hence

$$\phi_k(x^\frown k) = \phi^k(x)^\frown \vec{u}_k{}^\frown v_{j_q}{}^\frown v_{j_{q+1}}{}^\frown \cdots {}^\frown v_{j_{r-1}}.$$

It is enough to show now that all vectors to the right of $\vec{u}_k$ in the expression above are bounded by $\max(\tau_k^0)$. This is equivalent to show that either $q = r$ or $\max(v_{j_q}) \leq \max(\tau_k^0)$. Remember that $\max(v_\xi) = \max(\tau_\xi^1)$ for any $\xi$. By the definition[9] of $q$, one of the following two cases must hold:

Case 1: $\max(\tau_{j_q}^1) < \max(\tau_k)$. In this case, since $k = j_r$ and $q \leq r$ we have that

$$\max(\tau_{j_q}^1) \geq \max(\tau_{j_r}^1) = \max(\tau_k^1)$$

From the two inequalities above we conclude that $\max(\tau_k^1) < \max(\tau_k)$, hence $\max(\tau_k) = \max(\tau_k^0)$. Therefore $\max(\tau_{j_q}^1) < \max(\tau_k) = \max(\tau_k^0)$ as we wanted to prove.

Case 2: $\max(\tau_{j_q}^1) \geq \max(\tau_k)$ and $j_q \leq k$. Now, $j_q \leq k$ implies that

$$\max(\tau_{j_q}^1) \leq \max(\tau_{j_q}) \leq \max(\tau_k)$$

hence actually $\max(\tau_{j_q}^1) = \max(\tau_k)$. If $\max(\tau_k) = \max(\tau_k^0)$ then we are done, so we suppose that $\max(\tau_k) = \max(\tau_k^1) > \max(\tau_k^0)$. We combine the two previous equations we get that

$$\max(\tau_{j_q}^1) = \max(\tau_k) = \max(\tau_k^1) = \max(\tau_{j_r}^1)$$

but this implies (by the way in which chose the order of the enumeration $\{j_1, \dots, j_p\}$ and the fact that $j_q \leq k = j_r$ assumed in Case 2) that $r \leq q$, hence $r = q$ as we wanted to prove. This finishes Case 2, and finishes the proof of initial case $w = (k)$ as well.

Now we suppose that Claim 3 holds for $w$, we fix $\xi \leq k$ and we shall prove that Claim 3 holds for $w^\frown \xi$ as well. If $r < q(\xi)$ then $\phi_k(x^\frown w^\frown \xi) = \phi_k(x^\frown w)$ and we apply directly the inductive hypothesis. Hence, we suppose that $r \geq q(\xi)$ and therefore

$$(\clubsuit) \ \phi_k(x^\frown w^\frown \xi) = \phi^k(x^\frown w)^\frown \vec{u}_k{}^\frown v_{j_{q(\xi)}}{}^\frown v_{j_{q(\xi)+1}}{}^\frown \cdots {}^\frown v_{j_{r-1}}.$$

On the other hand,

$$\phi^k(x^\frown w) = \phi_k(x^\frown w)^\frown \breve{u}_k{}^\frown 0^{l'_k}$$

[9] It should be noticed that since we suppose $r \geq q$ we cannot have $q = p + 1$, so the minimum that defines $q$ is actually attained at $q$.

# Proof of the max function theorem



so applying the inductive hypothesis to $\phi_k(x^\frown w)$, we get that

$$\phi^k(x^\frown w) = \phi^k(x)^\frown \vec{u}_k{}^\frown w'$$

with $\max(w') \leq \max(\tau_k^0)$. Looking back at the expression (♣) above, it is enough to show that all members of that expression to the right of $\phi^k(x^\frown w)$ are bounded by $\max(\tau_k^0)$. This is equivalent to prove that either $r = q(\xi)$ or $\max(\tau^1_{j_{q(\xi)}}) = \max(v_{j_{q(\xi)}}) \leq \max(\tau_k^0)$. Let now $q = q(\xi)$. We distinguish two cases:

Case 1: $\max(\tau^1_{j_q}) < \max(\tau_k)$. In this case, since $k = j_r$ and we supposed that $q \leq r$ we have that

$$\max(\tau^1_{j_q}) \geq \max(\tau^1_{j_r}) = \max(\tau^1_k)$$

From the two inequalities above we conclude that $\max(\tau_k^1) < \max(\tau_k)$, hence $\max(\tau_k) = \max(\tau_k^0)$. Therefore $\max(\tau^1_{j_q}) < \max(\tau_k) = \max(\tau_k^0)$ as we wanted to prove.

Case 2: $\max(\tau^1_{j_q}) \geq \max(\tau_k)$. Since $\xi \leq k$ this implies that $\max(\tau^1_{j_q}) \geq \max(\tau_k) \geq \max(\tau_\xi)$. By the definition[9] of $q = q(\xi)$, this further implies that $j_q \leq \xi$. Now, $j_q \leq k$ implies that

$$\max(\tau^1_{j_q}) \leq \max(\tau_{j_q}) \leq \max(\tau_k)$$

hence actually $\max(\tau^1_{j_q}) = \max(\tau_k)$. If $\max(\tau_k) = \max(\tau_k^0)$ then we are done, so we suppose that $\max(\tau_k) = \max(\tau_k^1) > \max(\tau_k^0)$. We combine the previous equations and we get that

$$\max(\tau^1_{j_q}) = \max(\tau_k) = \max(\tau^1_k) = \max(\tau^1_{j_r})$$

but this implies (by the way in which chose the order of the enumeration $\{j_1, \dots, j_p\}$ and the fact that $j_q \leq \xi \leq k = j_r$ that we noticed above) that $r \leq q$, hence $r = q$ as we wanted to prove. This finishes Case 2, and finishes the proof of Claim 3 as well.

We fix $k < n$ and we shall prove that if $Y \subset n^{<\omega}$ is a set of type $[k]$, then $\phi(Y)$ is a set of type $\tau_k$. This will finish the proof of the theorem because, if $\phi$ was not a normal embedding, we can get a normal embedding by composing with a nice embedding using Theorem 1.3.

If $\tau_k$ is a chain type, then the fact that $\phi(Y)$ has type $\tau_k$ follows immediately from Claim 2. So suppose that $\tau_k$ is a comb type, $k = j_r$, and $Y = \{y_1, y_2, y_3, \dots, \}$. If we look at the inductive definition of $\phi$, and consider the case the case when $z = x^\frown k$ and $k = j_r$, notice that then $r \geq q$ by the definition of $q$ since $j_r = k \leq k$, and we can write

$$\phi(z) = \phi_k(z)^\frown v_{j_r}{}^\frown v_{j_{r+1}}^\frown \cdots {}^\frown v_p$$

where $\max(v_{j_t}) \leq \max(v_{j_r}) = \max(v_k)$ for all $t = r+1, \dots, p$. If we apply this to $z = y_i$ we can write that

$$(*)\quad \phi(y_i) = \phi_k(y_i)^\frown v_k^\frown w_i$$

where $\max(w_i) \leq \max(v_k)$. On the other hand, Claim 3 provides the fact that

$$(**)\quad \phi_k(y_{i+1}) = \phi^k(y_i)^\frown \vec{u}_k{}^\frown w_i' = \phi_k(y_i)^\frown \breve{u}_k{}^\frown 0^\zeta{}^\frown \vec{u}_k{}^\frown w_i'$$

where $\max(w_i) \leq \max(u_i)$. Remember that in the inductive definition of $\phi$, the number $\zeta$ of 0's above was chosen so that the length of $\phi_k(y_i)^\frown \breve{u}_k{}^\frown 0^\zeta$ is larger than



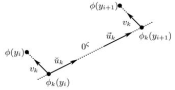



FIGURE 3. The structure of $\phi(Y)$ as a $\tau_k$-set

the length of $\phi(y_i)$. The expressions $(*)$ and $(**)$ together yield that $\phi(Y)$ is a set of type $\tau_k$ with underlying chain $\{\phi_k(y_i) : i < \omega\}$, as it is shown in Figure 3. □

COROLLARY 2.2. *If $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ is a normal embedding, then $\max(\tau) \leq \max(\tau')$ implies that $\max(\bar\phi\tau) \leq \max(\bar\phi\tau')$.*

COROLLARY 2.3. *If $\{S_i : i \in n\}$ are pairwise disjoint sets of types in $m^{<\omega}$, then $\{\Gamma_{S_i} : i \in n\}$ is an $n$-gap.*

PROOF. The intersection of two sets of different types is finite, so it is clear that the ideals are mutually orthogonal. We have to prove that they cannot be separated. After reordering if necessary, we can find types $\tau_i \in S_i$ such that $\max(\tau_0) \leq \max(\tau_1) \leq \cdots \leq \max(\tau_{n-1})$. By Theorem 2.1, there is a normal embedding $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ such that $\bar\phi[i] = \tau_i$. Finally, use Lemma 0.23. □

We can provide now our first example of a minimal analytic $n$-gap:

COROLLARY 2.4. *Let $\mathcal{M}_i$ be the set of all types $\tau$ in $n^{<\omega}$ such that $\max(\tau) = i$. The $n$-gap $\mathcal{M} = \{\Gamma_{\mathcal{M}_i} : i < n\}$ in $n^{<\omega}$ is a minimal $n$-gap.*

PROOF. Suppose that $\Gamma \leq \mathcal{M}$ and we must show that $\mathcal{M} \leq \Gamma$. By Theorem 0.25, we can suppose that $\Gamma = \{\Gamma_{S_i} : i < n\}$ is a standard gap in $n^{<\omega}$. That is, there is a permutation $\varepsilon : n \longrightarrow n$ such that $[i] \in S_{\varepsilon(i)}$. By Theorem 1.3, there is a normal embedding $\phi : n^{<\omega} \longrightarrow n^{<\omega}$ such that $\tau \in S_i$ if and only if $\bar\phi\tau \in \mathcal{M}_i$. In particular, $\bar\phi[i] \in \mathcal{M}_{\varepsilon(i)}$, so $\max \bar\phi[i] = \varepsilon(i)$. Since

$$\max[0] \leq \max[1] \leq \cdots \leq \max[n-1],$$

Corollary 2.2 implies that

$$\max \bar\phi[0] \leq \max \bar\phi[1] \leq \cdots \leq \max \bar\phi[n-1],$$

so $\varepsilon(0) \leq \varepsilon(1) \leq \cdots$ which implies that $\varepsilon$ is the identity permutation. Moreover, we claim that $\Gamma = \mathcal{M}$. For pick $\tau \in \mathcal{M}_i$. Then $\max(\tau) = \max[i]$, so $\max(\bar\phi\tau) = \max(\bar\phi[i]) = i$ which implies that $\bar\phi\tau \in \mathcal{M}_i$, hence $\tau \in S_i$. This shows that $\mathcal{M}_i \subset S_i$ for every $i$. Since the union of the sets $\mathcal{M}_i$ gives all types in $n^{<\omega}$ this actually implies that $\mathcal{M}_i = S_i$ for every $i < n$. □

For a permutation $\delta : n \longrightarrow n$, let us denote by $\mathcal{M}^\delta = \{\Gamma_{\mathcal{M}_{\delta(i)}} : i < n\}$ the $\delta$-permutation of $\mathcal{M}$. The minimal gaps $\mathcal{M}^\delta$ are characterized by their *extreme asymmetry* in the following sense:

COROLLARY 2.5. *The minimal $n$-gap $\mathcal{M}^\delta$ has the following two properties:*

(1) *$\mathcal{M}$ is dense.*

# Domination

**Definition**

We say that the type $\tau$ dominates the type $\sigma$ if

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Examples:

- $[{}^{02}{}_{1}{}^{3}{}_{2}]$ dominates $[02]$
- $[{}^{023}{}_{12}]$ does not dominate $[02]$
- $[{}^{1}{}_{5}]$ does not dominate $[{}^{0}{}_{2}]$.

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**Theorem**

For $\sigma, \tau$ types in $m^{<\omega}$, TFAE

1. $\tau$ dominates $\sigma$,
2. There exists a morphsim $f : \mathfrak{T}_2 \longrightarrow \mathfrak{T}_m$ such that
   - $f[0] = \sigma$,
   - $f\upsilon = \tau$ for all other $\upsilon \in \mathfrak{T}_2$.

# Proof of the domination theorem



case: it can be taken a top-comb type with $\max((\tau')^1) = k-1$. In this way we reduce the general case to the first case. □

If $\phi$ satifies the conditions of Lemma 3.3 we shall say that $\phi$ collapses below $k$ (or that $\phi$ collapses up to $k-1$) into a chain of type $\sigma$. The fact that in condition (1) of Lemma 3.3, the maximum of $\tau$ is attained in $\tau^1$ is important, for consider the following example: We can construct a normal embedding $\phi : 3^{<\omega} \longrightarrow 2^{<\omega}$ such that for every $x$, $\phi(x^\frown 2) \geq \phi(x)^\frown 1$, and $\phi(x^\frown i)$ equals $\phi(x)$ followed by a finite sequence of 0's when $i = 0, 1$. Such an embedding can be constructed inductively so that $x \prec y$ implies $|\phi(x)| < |\phi(y)|$. Notice that $\bar\phi[{}^0{}_1{}_2] = [01]$ but $\phi$ does not collapse below 3.

## 4. Domination

The notion of top-comb introduced in Definition 3.2 and illustrated in Figure 4 is going to be crucial in this section. The key property now will be the following:

LEMMA 4.1. *Let $\tau$ be a top-comb type and let $(u, v)$ be a rung of type $\tau$. If $w$ is such that $\max(w) \leq \max(\tau^1)$ and $|v^\frown w| < |u|$, then $(u, v^\frown w)$ is also a rung of type $\tau$.*

PROOF. Straightforward. Just look at the left-hand side of Figure 4. □

DEFINITION 4.2. We say that a type $\tau$ dominates another type $\sigma$, and we will write $\tau \gg \sigma$, if $\tau$ is a top-comb type and $\max(\tau^1) \geq \max(\sigma)$.

LEMMA 4.3. *Let $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ be a normal embedding, and let $\tau \in \mathfrak{T}_m$ be a type that dominates $\bar\phi\sigma$ for all $\sigma \in \mathfrak{T}_n$. Then, there exists a normal embedding $\psi : (n+1)^{<\omega} \longrightarrow m^{<\omega}$ such that $\bar\psi\sigma = \bar\phi\sigma$ if $\max(\sigma) < n$, and $\bar\psi\sigma = \tau$ if otherwise $\max(\sigma) = n$.*

PROOF. Let $m_0 = \max(\tau^1) + 1$. Without loss of generality we will suppose that $m = m_0$. We can do this because the domination hypothesis implies that all types $\bar\phi\sigma$ live in $m_0^{<\omega}$, and therefore we can find[12] $\phi_0 : n^{<\omega} \longrightarrow m_0^{<\omega}$ such that $\bar\phi_0\sigma = \bar\phi\sigma$ for all $\sigma$. Let $Y = \{y_0, y_1, \ldots\}$ be an infinite subset of $m^{<\omega}$ of type $\tau$, and let $b : n^{<\omega} \longrightarrow \{1, 2, 3, \ldots\}$ be a bijection such that $x \prec y$ if and only if $b(x) < b(y)$. If $x \in (n+1)^\omega \setminus n^{<\omega}$, there is a unique way to write $x$ in the form $x = u^\frown n^\frown v$ with $u \in (n+1)^{<\omega}$ and $v \in n^{<\omega}$, by splitting $x$ at the position of the last coordinate equal to $n$. Using this, we can define $\psi : (n+1)^{<\omega} \longrightarrow m^{<\omega}$ as

$$\begin{aligned} \psi(v) &= y_0{}^\frown\phi(v) \\ \psi(u^\frown n^\frown v) &= y_{b(u)}{}^\frown\phi(v) \end{aligned}$$

where $v \in n^{<\omega}$, $u \in (n+1)^{<\omega}$.

Claim 1: If $X \subset (n+1)^{<\omega}$ is a set of type $\sigma$ with $\max(\sigma) < n$, then $\psi(X)$ is a set of type $\sigma$.

Proof of Claim 1: This is clear, because $X$ must be either contained in either $n^{<\omega}$, in which case $\psi(X) = \phi(X)$, or $X$ is contained in a set of the form $\{u^\frown n^\frown v : v \in n^{<\omega}\}$ for some $v \in n^{<\omega}$, in which case $\psi(X) = \{y_{b(u)}{}^\frown x : x \in X\}$.

[12]One way to do this is to define $\phi_0(t) = (s_0', \ldots, s_k')$, where $\phi(t) = (s_0, \ldots, s_k)$, $s_i' = \min(s_i, m_0 - 1)$.



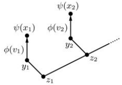



FIGURE 5. The set $\psi(X)$

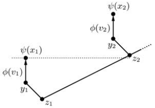



FIGURE 6. The set $\psi(X)$ after passing to a subsequence.

Claim 2: If $X \subset (n+1)^{<\omega}$ is a set of type $\sigma$, with $\max(\sigma) = n$, then $X$ contains an infinite subset $X'$ such that $\psi(X')$ has type $\tau$.

Proof of Claim 2: Let $X = \{x_1, x_2, \ldots\}$, and write $x_i = u_i^\frown n^\frown v_i$ in the form indicated above, with $v_i \in n^{<\omega}$. Since $X$ has type $\sigma$ with $\max(\sigma) = n$, we must have[13] $u_i \neq u_j$ for $i \neq j$. We have that $\phi(x_i) = y_{b(u_i)}{}^\frown\phi(v_i)$. By re-enumerating, let us suppose that $\phi(x_i) = y_i{}^\frown\phi(v_i)$ and remember that $\{y_1, y_2, \ldots\}$ has type $\tau$, so that the set $\psi(X)$ looks like in Figure 5. Let $z_i = y_{i+1} \wedge y_{i+2}$ the root nodes sitting on the chain below $Y$. By passing to a subsequence, we can suppose that $|\psi(x_i)| < |z_i|$ for all $i$, as illustrated in Figure 6. Once we do this, we claim that $\psi(X)$ has type $\tau$. We have to check that $(z_{i+1} \setminus z_i, \psi(x_i) \setminus z_i)$ is a rung of type $\tau$. We know that $(z_{i+1} \setminus z_i, y_i \setminus z_i)$ is a rung of type $\tau$, since $Y$ was of type $\tau$. Remember that $\psi(x_i) = y_i{}^\frown\phi(v_i)$, and we made an assumption at the beginning of the proof that $m = \max(\tau^1) \geq \max(\phi(v_i))$. We can apply Lemma 4.1 for $u = z_{i+1} \setminus z_i$, $v = y_i \setminus z_i$ and $w = \phi(x_i)$. □

THEOREM 4.4. *For $\{\tau_i : i \in n\} \subset \mathfrak{T}_m$ pairwise different, the following are equivalent:*

(1) *$\tau_k$ dominates $\tau_{k-1}$ for every $k = 1, \ldots, n-1$,*
(2) *there exists a normal embedding $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ such that $\bar\phi\sigma = \tau_{\max(\sigma)}$ for every $\sigma \in \mathfrak{T}_n$.*

PROOF. That (1) implies (2) follows from repeated applicacion of Lemma 4.3. We prove that (2) implies (1). As a first case, we prove the implication when $n = 2$ and $k = 1$. Thus, we have $\tau_0 \neq \tau_1$ and a normal embedding $\phi : 2^{<\omega} \longrightarrow m^{<\omega}$ such

[13]If we had $u_i = u_j$ for $i < j$, then the set $\{x_i, x_j\}$ would be equivalent to $\{v_i, v_j\} \subset n^{<\omega}$, but being $X$ of type $\sigma$, it is also equivalent to $\{v, u^\frown v\}$ for a rung $(u, v)$ of type $\sigma$, and $\max(\sigma) = n$.

# Proof of the domination theorem



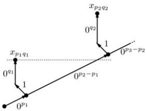



FIGURE 7. The nodes $x_{p_nq_n}$ in a sequence with $(\star)$.

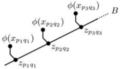



FIGURE 8. The nodes $\phi(x_{p_nq_n})$ as a set of type $\tau_1$ above the branch $B$.

that $\bar\phi[0] = \tau_0$ and $\bar\phi\sigma = \tau_1$ for every type $\sigma \neq [0]$ in $2^{<\omega}$. Notice that $\tau_1$ cannot be a chain type by Lemma 3.3. Consider the elements $x_{pq} = 0^p{}^\frown 1^\frown 0^q \in 2^{<\omega}$ (here $0^p$ means a sequence of $p$ many zeros). Notice that whenever $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$ are such that

$$(\star) \quad q_n + 1 < p_{n+1} - p_n,$$

the set $X = \{x_{p_1q_1}, x_{p_2q_2}, \cdots\}$ is of type $[^1{}_0]$, see Figure 7. Hence $\phi(X)$ is of (comb) type $\tau_1$, so it looks like in Figure 8. Let $B$ be the underlying branch of this set $\phi(X)$ that we can view in Figure 8, and we can formally define as

$$B = \{t : \exists i \ \ \forall j > i \ \ t < \phi(x_{p_jq_j})\}.$$

Claim A: The branch $B$ does not depend on the choice of the sequences $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$ with property $(\star)$ above. Proof of Claim A: Choose different sequences $p'_1 < p'_2 < \cdots$ and $q'_1 < q'_2 < \cdots$, and consider $X'$ and $B'$ the analogues of the set $X$ and the branch $B$ obtained from this new sequences of integers. Observe that $X$ and $X'$ can be alternated to produce a set of the form

$$Y = \{x_{p_{k_1}q_{k_1}}, x_{p'_{k_2}q'_{k_2}}, x_{p_{k_3}q_{k_3}}, x_{p'_{k_4}q'_{k_4}}, \cdots\}$$

and the sequence $k_1 < k_2 < \cdots$ can be chosen to grow fast enough so that property $(\star)$ is satisfied, and $Y$ is again a set of type $[^1{}_0]$. Then $\phi(Y)$ is a set of type $\tau_1$ again of the form represented in Figure 8 with underlying branch $B_Y$. But $\phi(Y)$ contains both an infinite subsequence contained in $\phi(X)$ and an infinite subsequence contained in $\phi(X')$. This implies that the equality of the underlying branches $B = B_Y = B'$, and finishes the proof of Claim A.



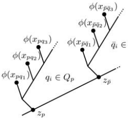



FIGURE 9. Sets of type $\tau_0$ over a $\tau_1$-set

Now, for $p, q < \omega$ let $z_{pq} = \max\{t \in B : t < \phi(x_{pq})\}$. We distinguish two cases:

Case 1: There exists $p < \omega$ and $q_1 < q_2 < \cdots$ such that $z_{pq_0} < z_{pq_1} < z_{pq_2} < \cdots$. In this case, $\{x_{pq_1}, x_{pq_2}, \ldots\}$ has type $[0]$, hence $Z = \{\phi(x_{pq_1}), \phi(x_{pq_2}), \ldots\}$ has type $\tau_0$. But each $\phi(x_{pq_r})$ goes out from the chain $B$ at the node $z_{pq_r}$, so these nodes $\phi(x_{pq_r})$ of the set $Z$ are displayed exactly in the same way as shown in Figure 8 (with now $p = p_1 = p_2 = \cdots$). We argue now that actually $Z$ contains a subsequece of type $\tau_1$, and this derives a contradiction since we said that $Z$ has type $\tau_0$ and we supposed that $\tau_0 \neq \tau_1$. The point is that each node $x_{pq_r}$ is a member of some sequence $\{x_{p'_nq'_n}\}$ having property $(\star)$, so each node $\phi(x_{pq_r})$ is a node of some set of type $\tau_1$ with underlying branch $B$. Thus, for high enough $t \in B$, the pair $(t \setminus z_{pq_r}, \phi(x_{pq_r}) \setminus z_{pq_r})$ is a rung of type $\tau_1$. In this way, we can construct a subsequence of $Z$ of type $\tau_1$ as desired.

Case 2: For every $p$ there exists an infinite set $Q_p \subset \omega$ such that $z_{pq} = z_{pq'}$ for all $q, q' \in Q_p$. We denote $z_p = z_{pq}$, $q \in Q_p$. We can also suppose[14] that $\phi(x_{pq}) > z_p$ for all $q \in Q_p$. The set $Y_p = \{\phi(x_{pq}) : q \in Q_p\}$ is now a set of type $\tau_0$ because it is the image under $\phi$ of a set of type $[0]$. Moreover, all elements of $Y_p$ are above $z_p$. The situation is illustrated in Figure 9. Similarly as in Case 1, we know that each $\phi(x_{pq})$ is an element of a set of type $\tau_1$ with underlying branch $B$, so $(t \setminus z_p, \phi(x_pq) \setminus z_p)$ is a rung of type $\tau_1$ for every $p$, $q$ and high enough $t \in B$. We prove now that $\tau_1$ dominates $\tau_0$. Pick $q_1 < q_2$ in $Q_p$. We have that $\max(\tau_1^1) = \max(\phi(x_{pq_2}) \setminus z_p)$, but since $Y_p$ is of type $\tau_0$,

$$\max(\phi(x_{pq_2}) \setminus z_p) \geq \max(\phi(x_{pq_2}) \setminus \phi(x_{pq_1})) = \max(\tau_0),$$

which proves that $\max(\tau_1^1) \geq \max(\tau_0)$. Finally, we prove that $\tau_1$ is a top-comb type. We know that $(u, v) = (t \setminus z_p, \phi(x_{pq_1}) \setminus z_p)$ is a rung of type $\tau_1$ for some high enough $t$. Let $h$ be the length of the last critical step of $u$. That is, if $u = u_1{}^\frown \cdots {}^\frown u_n$ with $u_i \in W_{k_i}$ as in Definition 3.6, let $h = |z_p{}^\frown u_1^\frown \cdots {}^\frown u_{n-1}|$. We can pick $q_3 \in Q_p$ such that $|\phi(x_{pq_3})| > h$. Then $(u', v') = (t \setminus z_p, \phi(x_{pq_3}) \setminus z_p)$ must be again a rung of type $\tau_1$ for high enough $t$, and we made sure that this rung satisfies the top-comb condition as illustrated in Figure 4.

[14] $\phi$ is one-to-one so there is at most one $q$ such that $\phi(x_{pq}) = z_p$.

# Proof of the domination theorem



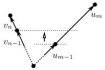



FIGURE 10. rung of a top$^2$ comb type

That finished the proof of the case when $n = 2$ and $k = 1$. For the general case, consider a normal embedding $\psi : 2^{<\omega} \longrightarrow n^{<\omega}$ given by $\psi(i_0, \dots, i_p) = (k-1+i_0, \dots, k-1+i_p)$. Then we can apply the case when $n = 2$ and $k = 1$ to $\phi' = \phi \circ \psi$, $\tau'_0 = \tau_{k-1}$ and $\tau'_1 = \tau_k$. $\square$

COROLLARY 4.5. *If $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ is a normal embedding, $\tau \gg \tau'$ and $\bar\phi\tau \neq \bar\phi\tau'$, then $\bar\phi\tau \gg \bar\phi\tau'$.*

COROLLARY 4.6. *Let $\phi : n^{<\omega} \longrightarrow m^{<\omega}$ be a normal embedding, $\tau$ a top-comb type with $\max(\tau^1) = k$, and suppose that $\bar\phi$ is not constant equal to $\bar\phi\tau$ on the set of types of maximum at most $k$. Then $\bar\phi\tau$ is a top-comb type.*

COROLLARY 4.7. *Let $\mathcal{M}$ be the minimal $n$-gap of Corollary 2.4, and let $\{S_i : i < n\}$ be pairwise disjoint nonempty families of types in $m^{<\omega}$. The following are equivalent:*

(1) $\mathcal{M} \leq \{\Gamma_{S_i} : i < n\}$,
(2) *we can pick $\tau_i \in S_i$ such that $\tau_0 \ll \tau_1 \ll \cdots \ll \tau_{n-1}$.*

## 5. Subdomination

When we remove from domination the condition of being a top-comb, we obtain the notion of subdomination.

DEFINITION 5.1. We say that a type $\tau$ subdominates another type $\sigma$, and we will write $\tau \gg_* \sigma$, if $\tau = (\tau^0, \tau^1)$ is a comb type which is not top-comb, and $\max(\tau^1) \geq \max(\sigma)$.

Lemma 4.3 says that when a type dominates $\tau$ the range of a normal embedding $\phi$, then it is possible to define a new normal embedding $\psi$ whose range equals the range of $\phi$ plus the type $\tau$. In this section, we shall see that if $\tau$ only subdominates the range of $\phi$, then we can find a normal embedding $\psi$ whose range contains the range of $\phi$, plus the type $\tau$, plus maybe at most five more types, which are formally described in Definition 5.2 and illustrated in Figures 11 and 12.

DEFINITION 5.2. Given a comb type $\tau$ which is not top-comb, we associate to it other comb types:

(1) $\mathfrak{k}(\tau)$ is exactly equal to $\tau$ except that the last element of $\tau^1$ is moved to the penultimate position in the order $\vartriangleleft$ in order to make $\mathfrak{k}(\tau)$ a comb type. For example, if $\tau = \left[{}^{23}{}_{167}\right]$, then $\mathfrak{k}(\tau) = \left[{}^{2}{}_{16}{}^{3}{}_{7}\right]$.

# Illustrative proof

We shall sketch the proof of the results announced at the beginning:

**Theorem 1**

If $\mathbf{\Gamma_0}, \ldots, \mathbf{\Gamma_{n-1}}$ is an analytic $n$-gap, then $\exists M \subset N$ :

- $\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M$ form a 2-gap.
- $\mathbf{\Gamma_k}|_M = \emptyset$ for all but at most 6 many of the remaining $k$

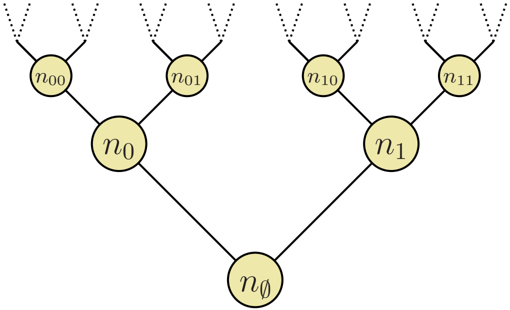
**Theorem 2**

If $\mathbf{\Gamma_0}, \ldots, \mathbf{\Gamma_{n-1}}$ is an analytic $n$-gap, then $\exists M \subset N$ and $i < j < n$ :

- $\mathbf{\Gamma_i}|_M, \mathbf{\Gamma_j}|_M$ form a 2-gap.
- $\mathbf{\Gamma_k}|_M = \emptyset$ for all other $k$

# Sketch of some proofs

Step 1: We apply our general theorem to the gap $\{\mathbf{\Gamma_0}, \mathbf{\Gamma_1}\}$




$$[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{\Gamma_1}$$

$$[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1} \text{ or } \mathbf{\Gamma_0}$$

$$[\mathbf{01}]$$

$$[{}^{\mathbf{0}}{}_{\mathbf{1}}]$$

$$[{}^{\mathbf{1}}{}_{\mathbf{0}}]$$

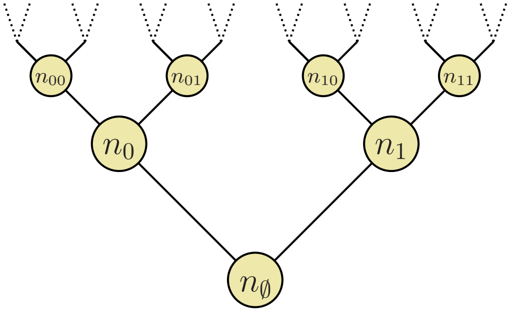
$$[{}^{\mathbf{01}}{}_{\mathbf{1}}]$$

$$[{}^{\mathbf{1}}{}_{\mathbf{01}}]$$

$$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}]$$

# Sketch of some proofs

Step 1: We apply our general theorem to the gap $\{\mathbf{\Gamma_0}, \mathbf{\Gamma_1}\}$




$[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0}$

$[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1}$

$[\mathbf{01}]$

$[{}^{\mathbf{0}}{}_{\mathbf{1}}]$

$[{}^{\mathbf{1}}{}_{\mathbf{0}}]$

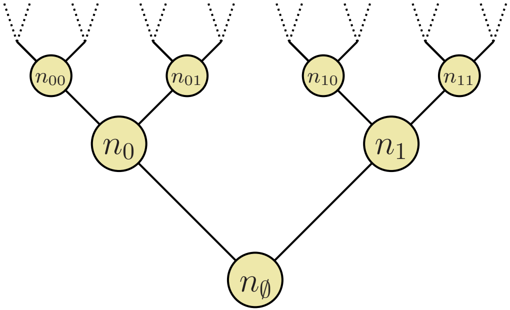
$[{}^{\mathbf{01}}{}_{\mathbf{1}}]$

$[{}^{\mathbf{1}}{}_{\mathbf{01}}]$

$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}]$ $\quad \Gamma_f$ or $X$

# Sketch of some proofs

Step 2: Apply the Ramsey theorem




$$[\mathbf{0}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{0}}$$

$$[\mathbf{1}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{1}}$$

$$[\mathbf{01}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{a}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{0}}{}_{\mathbf{1}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{b}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{1}}{}_{\mathbf{0}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{c}} \text{ or } \mathbf{X}$$

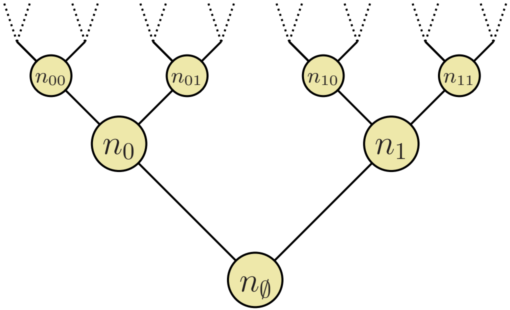
$$[{}^{\mathbf{01}}{}_{\mathbf{1}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{d}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{1}}{}_{\mathbf{01}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{e}} \text{ or } \mathbf{X}$$

$$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{f}} \text{ or } \mathbf{X}$$

# Sketch of some proofs

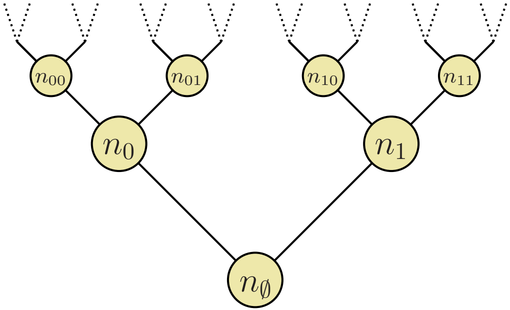
Step 2: Apply the Ramsey theorem and we have Theorem 1!




$$[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0}$$
$$[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1}$$
$$[\mathbf{01}] \longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X}$$
$$\left[{}^{\mathbf{0}}{}_{\mathbf{1}}\right] \longrightarrow \mathbf{\Gamma_b} \text{ or } \mathbf{X}$$
$$\left[{}^{\mathbf{1}}{}_{\mathbf{0}}\right] \longrightarrow \mathbf{\Gamma_c} \text{ or } \mathbf{X}$$
$$\left[{}^{\mathbf{01}}{}_{\mathbf{1}}\right] \longrightarrow \mathbf{\Gamma_d} \text{ or } \mathbf{X}$$
$$\left[{}^{\mathbf{1}}{}_{\mathbf{01}}\right] \longrightarrow \mathbf{\Gamma_e} \text{ or } \mathbf{X}$$
$$\left[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}\right] \longrightarrow \mathbf{\Gamma_f} \text{ or } \mathbf{X}$$

# Sketch of some proofs

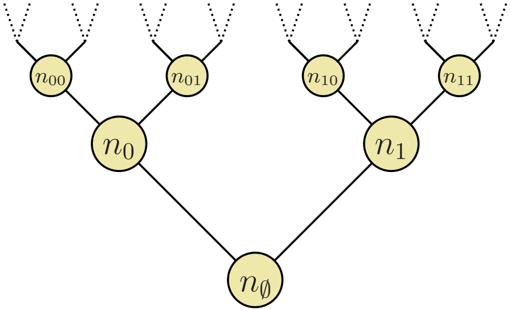
Now we go for Theorem 2.




$$[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0}$$
$$[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1}$$
$$[\mathbf{01}] \longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X}$$
$$[{}^{\mathbf{0}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_b} \text{ or } \mathbf{X}$$
$$[{}^{\mathbf{1}}{}_{\mathbf{0}}] \longrightarrow \mathbf{\Gamma_c} \text{ or } \mathbf{X}$$
$$[{}^{\mathbf{01}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_d} \text{ or } \mathbf{X}$$
$$[{}^{\mathbf{1}}{}_{\mathbf{01}}] \longrightarrow \mathbf{\Gamma_e} \text{ or } \mathbf{X}$$
$$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_f} \text{ or } \mathbf{X}$$

# Sketch of some proofs

Observe that $[{}^{0}{}_{1}]$ dominates $[0]$,




$[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0}$

$[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1}$

$[\mathbf{01}] \longrightarrow \mathbf{\Gamma_a}$ or $\mathbf{X}$

$[{}^{\mathbf{0}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_b}$ or $\mathbf{X}$

$[{}^{\mathbf{1}}{}_{\mathbf{0}}] \longrightarrow \mathbf{\Gamma_c}$ or $\mathbf{X}$

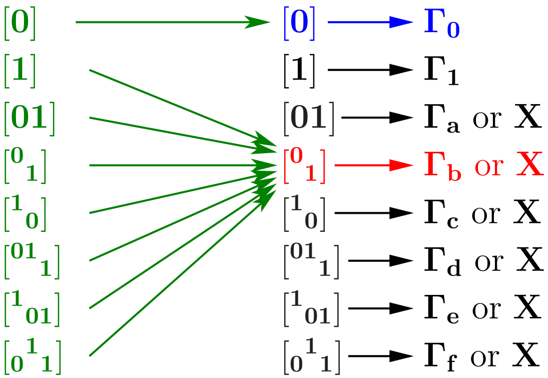
$[{}^{\mathbf{01}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_d}$ or $\mathbf{X}$

$[{}^{\mathbf{1}}{}_{\mathbf{01}}] \longrightarrow \mathbf{\Gamma_e}$ or $\mathbf{X}$

$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_f}$ or $\mathbf{X}$

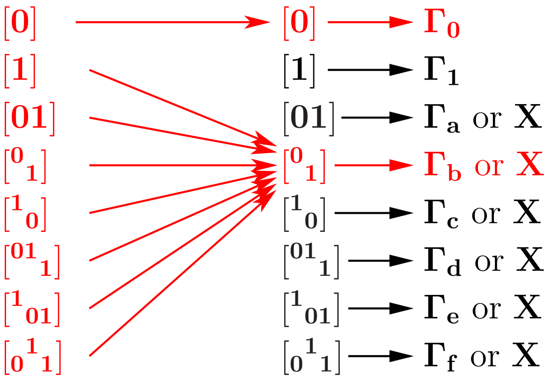
# Sketch of some proofs

Observe that $[{}^{0}{}_{1}]$ dominates $[0]$, so we have $u : 2^{<\omega} \longrightarrow 2^{<\omega}$



# Sketch of some proofs

Observe that $[{}^{0}{}_{1}]$ dominates $[0]$, So if $b \neq 0$ we are done.



# Sketch of some proofs

Observe that $[^0{}_1]$ dominates $[0]$, So if $b \neq 0$ we are done.

$$
\begin{aligned}
&[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0}\\
&[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1}\\
&[\mathbf{01}] \longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X}\\
&[^{\mathbf{0}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X}\\
&[^{\mathbf{1}}{}_{\mathbf{0}}] \longrightarrow \mathbf{\Gamma_c} \text{ or } \mathbf{X}\\
&[^{\mathbf{01}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_d} \text{ or } \mathbf{X}\\
&[^{\mathbf{1}}{}_{\mathbf{01}}] \longrightarrow \mathbf{\Gamma_e} \text{ or } \mathbf{X}\\
&[_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_f} \text{ or } \mathbf{X}
\end{aligned}
$$

# Sketch of some proofs

The same argument works for these other types.

$$[\mathbf{0}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{0}}$$

$$[\mathbf{1}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{1}}$$

$$[\mathbf{01}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{a}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{0}}{}_{\mathbf{1}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{0}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{1}}{}_{\mathbf{0}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{c}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{01}}{}_{\mathbf{1}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{d}} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{1}}{}_{\mathbf{01}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{e}} \text{ or } \mathbf{X}$$

$$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] \longrightarrow \boldsymbol{\Gamma}_{\mathbf{f}} \text{ or } \mathbf{X}$$

# Sketch of some proofs

The same argument works for these other types.

$$[\mathbf{0}] \longrightarrow \mathbf{\Gamma_0}$$

$$[\mathbf{1}] \longrightarrow \mathbf{\Gamma_1}$$

$$[\mathbf{01}] \longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{0}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{1}}{}_{\mathbf{0}}] \longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{01}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X}$$

$$[{}^{\mathbf{1}}{}_{\mathbf{01}}] \longrightarrow \mathbf{\Gamma_e} \text{ or } \mathbf{X}$$

$$[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] \longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X}$$

# Sketch of some proofs

But these types also dominate [1].

$$
\begin{array}{lcl}
[\mathbf{0}] & \longrightarrow & \mathbf{\Gamma_0} \\
[\mathbf{1}] & \longrightarrow & \mathbf{\Gamma_1} \\
[\mathbf{01}] & \longrightarrow & \mathbf{\Gamma_a} \text{ or } \mathbf{X} \\
[{}^{\mathbf{0}}{}_{\mathbf{1}}] & \longrightarrow & \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{0}}] & \longrightarrow & \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{01}}{}_{\mathbf{1}}] & \longrightarrow & \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{01}}] & \longrightarrow & \mathbf{\Gamma_e} \text{ or } \mathbf{X} \\
[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] & \longrightarrow & \mathbf{\Gamma_0} \text{ or } \mathbf{X}
\end{array}
$$

# Sketch of some proofs

But these types also dominate [1]. So if they go to $\mathbf{\Gamma_0}$, we are done.

$$
\begin{aligned}
[\mathbf{0}] &\longrightarrow \mathbf{\Gamma_0} \\
[\mathbf{1}] &\longrightarrow \mathbf{\Gamma_1} \\
[\mathbf{01}] &\longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X} \\
[{}^{\mathbf{0}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{0}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{01}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{01}}] &\longrightarrow \mathbf{\Gamma_e} \text{ or } \mathbf{X} \\
[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X}
\end{aligned}
$$

# Sketch of some proofs

But these types also dominate [1]. So if they go to $\mathbf{\Gamma_0}$, we are done.

$$
\begin{aligned}
[\mathbf{0}] &\longrightarrow \mathbf{\Gamma_0} \\
[\mathbf{1}] &\longrightarrow \mathbf{\Gamma_1} \\
[\mathbf{01}] &\longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X} \\
[{}^{\mathbf{0}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{0}}] &\longrightarrow \mathbf{X} \\
[{}^{\mathbf{01}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{01}}] &\longrightarrow \mathbf{\Gamma_e} \text{ or } \mathbf{X} \\
[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{X}
\end{aligned}
$$

# Sketch of some proofs

So far, we isolated at most four families.

$$
\begin{array}{lcl}
[\mathbf{0}] & \longrightarrow & \boldsymbol{\Gamma}_{\mathbf{0}} \\
[\mathbf{1}] & \longrightarrow & \boldsymbol{\Gamma}_{\mathbf{1}} \\
[\mathbf{01}] & \longrightarrow & \boldsymbol{\Gamma}_{\mathbf{a}} \text{ or } \mathbf{X} \\
[{}^{\mathbf{0}}{}_{\mathbf{1}}] & \longrightarrow & \boldsymbol{\Gamma}_{\mathbf{0}} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{0}}] & \longrightarrow & \mathbf{X} \\
[{}^{\mathbf{01}}{}_{\mathbf{1}}] & \longrightarrow & \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{01}}] & \longrightarrow & \boldsymbol{\Gamma}_{\mathbf{e}} \text{ or } \mathbf{X} \\
[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] & \longrightarrow & \mathbf{X}
\end{array}
$$

# Sketch of some proofs

Now look at the types $[^{1}{}_{01}]$ and $[0]$.

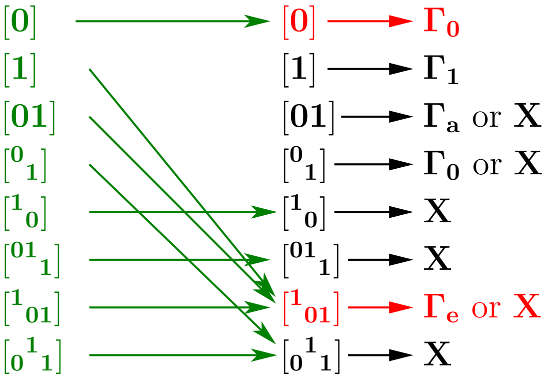
$$
\begin{array}{lcl}
\mathbf{[0]} & \longrightarrow & \mathbf{\Gamma_0} \\
\mathbf{[1]} & \longrightarrow & \mathbf{\Gamma_1} \\
\mathbf{[01]} & \longrightarrow & \mathbf{\Gamma_a} \text{ or } \mathbf{X} \\
\mathbf{[^{0}{}_{1}]} & \longrightarrow & \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
\mathbf{[^{1}{}_{0}]} & \longrightarrow & \mathbf{X} \\
\mathbf{[^{01}{}_{1}]} & \longrightarrow & \mathbf{X} \\
\mathbf{[^{1}{}_{01}]} & \longrightarrow & \mathbf{\Gamma_e} \text{ or } \mathbf{X} \\
\mathbf{[_{0}{}^{1}{}_{1}]} & \longrightarrow & \mathbf{X}
\end{array}
$$

# Sketch of some proofs

$\max[{}^{1}{}_{01}] = 1 \geq 0 = \max[0]$.

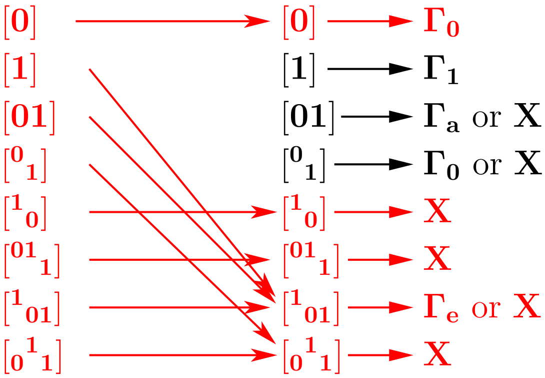
$$
\begin{tikzcd}
{[\mathbf{0}]} \arrow[r] & {[\mathbf{0}]} \arrow[r] & \mathbf{\Gamma_0} \\
{[\mathbf{1}]} \arrow[dddddr] & {[\mathbf{1}]} \arrow[r] & \mathbf{\Gamma_1} \\
{[\mathbf{01}]} & {[\mathbf{01}]} \arrow[r] & \mathbf{\Gamma_a} \text{ or } \mathbf{X} \\
{[{}^{\mathbf{0}}{}_{\mathbf{1}}]} & {[{}^{\mathbf{0}}{}_{\mathbf{1}}]} \arrow[r] & \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
{[{}^{\mathbf{1}}{}_{\mathbf{0}}]} & {[{}^{\mathbf{1}}{}_{\mathbf{0}}]} \arrow[r] & \mathbf{X} \\
{[{}^{\mathbf{01}}{}_{\mathbf{1}}]} & {[{}^{\mathbf{01}}{}_{\mathbf{1}}]} \arrow[r] & \mathbf{X} \\
{[{}^{\mathbf{1}}{}_{\mathbf{01}}]} & {[{}^{\mathbf{1}}{}_{\mathbf{01}}]} \arrow[r] & \mathbf{\Gamma_e} \text{ or } \mathbf{X} \\
{[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}]} & {[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}]} \arrow[r] & \mathbf{X}
\end{tikzcd}
$$

After some painful computation...



# Sketch of some proofs

After some painful computation... So if $e \neq 0$ we are done



## Sketch of some proofs

Now..

$$
\begin{aligned}
[\mathbf{0}] &\longrightarrow \mathbf{\Gamma_0} \\
[\mathbf{1}] &\longrightarrow \mathbf{\Gamma_1} \\
[\mathbf{01}] &\longrightarrow \mathbf{\Gamma_a} \text{ or } \mathbf{X} \\
[{}^{\mathbf{0}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{0}}] &\longrightarrow \mathbf{X} \\
[{}^{\mathbf{01}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{X} \\
[{}^{\mathbf{1}}{}_{\mathbf{01}}] &\longrightarrow \mathbf{\Gamma_0} \text{ or } \mathbf{X} \\
[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}] &\longrightarrow \mathbf{X}
\end{aligned}
$$

# Sketch of some proofs

Looking similarly at [01], we have...

$$
\begin{tikzcd}
{[\mathbf{0}]} \arrow[r] & {[\mathbf{0}]} \arrow[r] & \boldsymbol{\Gamma_0} \\
{[\mathbf{1}]} \arrow[dr] & {[\mathbf{1}]} \arrow[r] & \boldsymbol{\Gamma_1} \\
{[\mathbf{01}]} \arrow[r] & {[\mathbf{01}]} \arrow[r] & \boldsymbol{\Gamma_a} \text{ or } \mathbf{X} \\
{[{}^{\mathbf{0}}{}_{\mathbf{1}}]} \arrow[r] & {[{}^{\mathbf{0}}{}_{\mathbf{1}}]} \arrow[r] & \boldsymbol{\Gamma_0} \text{ or } \mathbf{X} \\
{[{}^{\mathbf{1}}{}_{\mathbf{0}}]} \arrow[r] & {[{}^{\mathbf{1}}{}_{\mathbf{0}}]} \arrow[r] & \mathbf{X} \\
{[{}^{\mathbf{01}}{}_{\mathbf{1}}]} \arrow[r] & {[{}^{\mathbf{01}}{}_{\mathbf{1}}]} \arrow[r] & \mathbf{X} \\
{[{}^{\mathbf{1}}{}_{\mathbf{01}}]} \arrow[r] & {[{}^{\mathbf{1}}{}_{\mathbf{01}}]} \arrow[r] & \boldsymbol{\Gamma_0} \text{ or } \mathbf{X} \\
{[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}]} \arrow[r] & {[{}_{\mathbf{0}}{}^{\mathbf{1}}{}_{\mathbf{1}}]} \arrow[r] & \mathbf{X}
\end{tikzcd}
$$

# Sketch of some proofs

So if $a \neq 0$ we are done,

$$
\begin{tikzcd}
{[0]} \arrow[r] & {[0]} \arrow[r] & \Gamma_0 \\
{[1]} \arrow[dr] & {[1]} \arrow[r] & \Gamma_1 \\
{[01]} \arrow[r] & {[01]} \arrow[r] & \Gamma_a \text{ or } X \\
{[{}^{0}{}_{1}]} \arrow[r] & {[{}^{0}{}_{1}]} \arrow[r] & \Gamma_0 \text{ or } X \\
{[{}^{1}{}_{0}]} \arrow[r] & {[{}^{1}{}_{0}]} \arrow[r] & X \\
{[{}^{01}{}_{1}]} \arrow[r] & {[{}^{01}{}_{1}]} \arrow[r] & X \\
{[{}^{1}{}_{01}]} \arrow[r] & {[{}^{1}{}_{01}]} \arrow[r] & \Gamma_0 \text{ or } X \\
{[{}_{0}{}^{1}{}_{1}]} \arrow[r] & {[{}_{0}{}^{1}{}_{1}]} \arrow[r] & X
\end{tikzcd}
$$

# Sketch of some proofs

So if $a \neq 0$ we are done, and otherwise as well.

$$
\begin{tikzcd}
{[0]} \arrow[r] & {[0]} \arrow[r] & \Gamma_0 \\
{[1]} \arrow[dr] & {[1]} \arrow[r] & \Gamma_1 \\
{[01]} \arrow[r] & {[01]} \arrow[r] & \Gamma_0 \text{ or } X \\
{[{}^{0}{}_{1}]} \arrow[r] & {[{}^{0}{}_{1}]} \arrow[r] & \Gamma_0 \text{ or } X \\
{[{}^{1}{}_{0}]} \arrow[r] & {[{}^{1}{}_{0}]} \arrow[r] & X \\
{[{}^{01}{}_{1}]} \arrow[r] & {[{}^{01}{}_{1}]} \arrow[r] & X \\
{[{}^{1}{}_{01}]} \arrow[r] & {[{}^{1}{}_{01}]} \arrow[r] & \Gamma_0 \text{ or } X \\
{[{}_{0}{}^{1}{}_{1}]} \arrow[r] & {[{}_{0}{}^{1}{}_{1}]} \arrow[r] & X
\end{tikzcd}
$$

## Another sample result

### Theorem 2

If $\mathbf{\Gamma_0}, \mathbf{\Gamma_1}, \mathbf{\Gamma_2}$ is an analytic 3-gap, then at least two of the following three hold: :

# Another sample result

**Theorem 2**

If $\mathbf{\Gamma_0}, \mathbf{\Gamma_1}, \mathbf{\Gamma_2}$ is an analytic 3-gap, then at least two of the following three hold: :

- $\exists M \subset N : \{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M\}$ form a 2-gap but $\mathbf{\Gamma_2}|_M = \emptyset$.

# Another sample result

**Theorem 2**

If $\mathbf{\Gamma_0}, \mathbf{\Gamma_1}, \mathbf{\Gamma_2}$ is an analytic 3-gap, then at least two of the following three hold: :

- $\exists M \subset N$ : $\{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M\}$ form a 2-gap but $\mathbf{\Gamma_2}|_M = \emptyset$.
- $\exists M \subset N$ : $\{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_2}|_M\}$ form a 2-gap but $\mathbf{\Gamma_1}|_M = \emptyset$.

# Another sample result

**Theorem 2**

If $\mathbf{\Gamma_0}, \mathbf{\Gamma_1}, \mathbf{\Gamma_2}$ is an analytic 3-gap, then at least two of the following three hold: :

- $\exists M \subset N$ : $\{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M\}$ form a 2-gap but $\mathbf{\Gamma_2}|_M = \emptyset$.
- $\exists M \subset N$ : $\{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_2}|_M\}$ form a 2-gap but $\mathbf{\Gamma_1}|_M = \emptyset$.
- $\exists M \subset N$ : $\{\mathbf{\Gamma_1}|_M, \mathbf{\Gamma_2}|_M\}$ form a 2-gap but $\mathbf{\Gamma_0}|_M = \emptyset$.

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**Theorem 2**

If $\mathbf{\Gamma_0}, \mathbf{\Gamma_1}, \mathbf{\Gamma_2}$ is an analytic 3-gap, then at least two of the following three hold: :

- $\exists M \subset N$ : $\{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_1}|_M\}$ form a 2-gap but $\mathbf{\Gamma_2}|_M = \emptyset$.
- $\exists M \subset N$ : $\{\mathbf{\Gamma_0}|_M, \mathbf{\Gamma_2}|_M\}$ form a 2-gap but $\mathbf{\Gamma_1}|_M = \emptyset$.
- $\exists M \subset N$ : $\{\mathbf{\Gamma_1}|_M, \mathbf{\Gamma_2}|_M\}$ form a 2-gap but $\mathbf{\Gamma_0}|_M = \emptyset$.

Proof: Just check it for each the 933 minimal analytic 3-gaps.