On existence of independent sets in partially ordered sets

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The strong sequences method was introduced by B. A. Efimov, as a useful method for proving famous theorems in dyadic spaces like: Marczewski theorem on cellularity, Shanin theorem on a calibre, Esenin-Volpin theorem, Erdös-Rado theorem and others.

Let T be an infinite set. Denote *the Cantor cube* by

$$D^T = \{p\colon p\colon T \to \{0,1\}\}.$$

For $s \subset T$, $i\colon s \to \{0,1\}$ it will be used the following notation

$$H^i_s = \{p \in D^T\colon p|s = i\}.$$

Efimov defined strong sequences in the subbase $\{H^i_{\{\alpha\}}\colon \alpha \in T\}$ of the Cantor cube and proved the following

Theorem (Efimov)

Let κ be a regular, uncountable cardinal number. In the space D^T there is not a strong sequence

$$(\{H^i_{\{\alpha\}} : \alpha \in v_\xi\}, \{H^i_{\{\beta\}} : \beta \in w_\xi\}) \;\; ; \;\; \xi < \kappa$$

such that $|w_\xi| < \kappa$ and $|v_\xi| < \omega$ for each $\xi < \kappa$.

Let X be a set, and $B \subset P(X)$ be a family of non-empty subsets of X closed with respect to finite intersections. Let S be a finite subfamily contained B. A pair (S, H), where $H \subseteq B$, will be called *connected* if $S \cup H$ is centered.

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Definition (Turzański)

A sequence (S_ϕ, H_ϕ); $\phi < \alpha$ consisting of connected pairs is called *a strong sequence* if $S_\lambda \cup H_\phi$ is not centered whenever $\lambda > \phi$.

Theorem (Turzański)

If for $B \subset P(X)$ there exists a strong sequence $S = (S_\phi, H_\phi); \phi < (\kappa^\lambda)^+$ such that $|H_\phi| \leq \kappa$ for each $\phi < (\kappa^\lambda)^+$ then there exists a strong sequence $(S_\phi, T_\phi); \phi < \lambda^+$, where $|T_\phi| < \omega$ for each $\phi < \lambda^+$

In 2008
J. Jureczko, M. Turzański,
From a Ramsey-Type Theorem To Independence,
Acta Universitatis Carolinae - Mathematica et Physica, vol. 49, no. 2, p. 47-55.

Definition

We say that a family of sets $\mathscr{S}$ fulfills condition (I) if for all $S_0, S_1, S_2 \in \mathscr{S}$, if $S_0 \cap S_1 = \emptyset$ and $S_0 \cap S_2 = \emptyset$ then either $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$ or $S_2 \subset S_1$.

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Definition

We say that a family of sets $\mathscr{S}$ fulfills condition $(T(\kappa))$ if for each set $U \in \mathscr{S}$ there is

$$|\{V \in \mathscr{S} : V \subset U\}| < \kappa$$

Definition

A family $\{(A^0_\xi, A^1_\xi)\colon \xi < \alpha\}$ of ordered pairs of subsets of X such that $A^0_\xi \cap A^1_\xi = \emptyset$ for $\xi < \alpha$ is called a weakly independent family (of length α) if for each $\xi, \zeta < \alpha$ with $\xi \neq \zeta$ we have $A^i_\xi \cap A^j_\zeta \neq \emptyset$, where $i, j \in \{0, 1\}$.

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Theorem

Let $\mathscr{S}$ be a family of sets which has the following properties:
(i) $\mathscr{S}$ fulfills condition (I);
(ii) $\mathscr{S}$ fulfills condition $(T(\kappa))$;
(iii) for each $U \in \mathscr{S}$ there is $X \setminus U \in \mathscr{S}$.
Then for each regular cardinal number κ such that $|\mathscr{S}| \geq \kappa > c(\mathscr{S})$ there exists a weakly independent family in $\mathscr{S}$ of cardinality κ.

Definition

A family of sets $\mathscr{S}$ is said to be binary if for each finite subfamily $\mathscr{M} \subset \mathscr{S}$ with $\bigcap \mathscr{M} = \emptyset$ there exist $A, B \in \mathscr{M}$ such that $A \cap B = \emptyset$.

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Definition

A family $\{(A_\xi, B_\xi) \colon \xi < \alpha\}$ of ordered pairs of subsets of X, such that $A_\xi \cap B_\xi = \emptyset$ for $\xi < \alpha$ is called an independent family (of length α) if for each finite subset $F \subset \alpha$ and each function $i \colon F \to \{-1, +1\}$ we have

$$\bigcap \{i(\xi) A_\xi : \xi \in F\} \neq \emptyset$$

(where $(+1)A_\xi = A_\xi, (-1)A_\xi = B_\xi$).

Corollary

Let X be a compact zero-dimensional space. Let $\mathscr{S}$ be a family consisting of clopen sets which has the following properties:
(i) $\mathscr{S}$ is a binary family;
(ii) $\mathscr{S}$ fulfills condition (I);
(iii) $\mathscr{S}$ fulfills condition $(T(\kappa))$;
(iv) for each $U \in \mathscr{S}$ the set $X \setminus U \in \mathscr{S}$.
Then for each regular cardinal number κ such that $|\mathscr{S}| \geq \kappa > c(\mathscr{S})$ there exists an independent family in $\mathscr{S}$ of cardinality κ.

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- If each of two elements in a set $A \subset X$ are compatible, then A is an *upper directed* set.
- A set A is κ- *upper directed* if every subset of X of cardinality less than κ has an upper bound, i.e. for each $B \subset X$ with $|B| < \kappa$ there exists $a \in A$ such that $(b,a) \in r$ for all $b \in B$.

Definition

Let (X, r) be a set with relation r.
A sequence $(S_\phi, H_\phi); \phi < \alpha$ where $S_\phi, H_\phi \subset X$ and S_ϕ is finite is called a strong sequence if
1^o $S_\phi \cup H_\phi$ is ω-upper directed
2^o $S_\beta \cup H_\phi$ is not ω-upper directed for $\beta > \phi$.

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- We say that (X, r) has $Q(\kappa)$-property iff for all $x, y \in X$ if $x \parallel y$ then
$$|\{z \in X\colon x \perp z \vee z \perp y\}| = \kappa.$$

- We say that $\mathscr{L} \subset X$ is a *chain* if any $a, b \in \mathscr{L}$ are comparable.

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- We say that a set $\mathscr{A} \subset X$ is called an *antichain* if any two distinct elements $a, b \in \mathscr{A}$ are incompatible.
- The minimal cardinal κ such that every antichain in X has size less than κ is *saturation of X* and denote it by $sat(X)$.

Definition

A sequence of ordered pairs $\{(x^0_\alpha, x^1_\alpha)\}$ where $x^0_\alpha \perp x^1_\alpha$ is said to be *an independent set* if for each finite set $F \subset \kappa$ and for each function $i\colon F \to \{0,1\}$ the set $\{x^{i(\alpha)}_\alpha \colon \alpha \in F\}$ is $\omega-$ upper directed.

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Theorem

Let κ be a regular cardinal number. Let (X, r) be a set with relation which has $A(\omega)$- and $Q(\omega)$-property. If $|X| = \kappa > sat(X)$ then there exists an independent set in X of cardinality κ.

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Theorem

Let κ be a regular number. Let (X, r) be a set with relation which has $A(\kappa)$- and $Q(\kappa)$-property. If $|X| = \kappa > sat(X)$ then there exists a $\kappa-$independent set in X of cardinality κ.

Definition

A cardinal κ is a *calibre* for X if κ is infinite and every set $A \in [X]^\kappa$ has a chain of size κ.

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A cardinal κ is a *precalibre* for X if κ is infinite and every set $A \in [X]^\kappa$ has ω -upper directed subset of cardinality κ.

- **Note** Each calibre is a precalibre but the inverse theorem is not true.

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- $$s = sup\{\kappa\colon \text{there exists a strong sequence in } X \text{ of the length } \kappa\}.$$

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Theorem

Let (X, r) be a set with relation r. Then each regular cardinal number $\tau > s$ is a precalibre for X.

Theorem

Let τ be a cardinal number. Let (X, r) be a set with relation and τ^+ be a precalibre of X. If $|X| > 2^\tau$, then there exists an independent set of cardinality τ^+.

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- $$i_\kappa = \sup\{|A| \colon A \text{ is a } \kappa\text{-independent set in } X\}.$$

Theorem

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Let (X, r) be a set with relation. Let τ be a regular cardinal number which is a precalibre for X. Then $i > \tau > s$.

Theorem

Let $\kappa \geq \omega$ and (X, r) be a set with relation of cardinality at least κ. If (X, r) has $A(\kappa)$- and $Q(\kappa)$-property then there exists a set $A \subset X$ of cardinality κ which is both a maximal κ-independent set and a maximal independent set.

Theorem

Let $\kappa \geq \omega$ and (X, r) be a set with relation of cardinality at least κ. If (X, r) has $A(\kappa)$- and $Q(\kappa)$-property then there exists a set $A \subset X$ of cardinality κ which is both a maximal κ-independent set and a maximal independent set.

Corollary

Let $\kappa \geq \omega$ and (X, r) be a set with relation of cardinality at least κ. If (X, r) has $A(\kappa)$- and $Q(\kappa)$-property, then $i_\kappa = i$.

Theorem

Let (X, r) *be a set with relation r. Then* $s \geq sat(X)$.

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Corollary

Let (X, r) *be a set with relation. Let* τ *be a precalibre of* X.
Then $i > \tau > s \geq sat(X)$.

Theorem

Let (X, r) be a set with relation r. Then $s \geq sat(X)$.

Corollary

Let (X, r) be a set with relation. Let τ be a precalibre of X. Then $i > \tau > s \geq sat(X)$.

Corollary

Let (X, r) be a set with relation. Let τ be a precalibre of X. Then $i_\kappa > \tau > s \geq sat(X)$.

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