Strongly Dominating Sets of Reals

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Outline

1. Motivation
2. Strongly dominating sets and the ideal $\mathcal{D}$
3. Analytic strongly dominating sets
4. Comparison of ideals $\mathcal{D}$ and l^0

Motivation

Lemma ([1], Goldstern, Repický, Shelah, Spinas)

For a Borel set $B \subseteq {}^{\omega}\omega$ *the following conditions are equivalent:*

- B *is strongly dominating.*
- *There is a Laver tree* p *such that* $[p] \subseteq B$.

Motivation

Lemma ([1], Goldstern, Repický, Shelah, Spinas)

For a Borel set $B \subseteq {}^{\omega}\omega$ the following conditions are equivalent:

- *B is strongly dominating.*
- *There is a Laver tree p such that $[p] \subseteq B$.*

Theorem ([2], Kechris)

For an analytic set $A \subseteq {}^{\omega}\omega$ the following conditions are equivalent:

- *A is unbounded in $({}^{\omega}\omega, \leq^{*})$.*
- *There exists a Miller tree q such that $[q] \subseteq A$.*

Strongly dominating sets and the ideal $\mathcal{D}$

Definition

For a set $A \subseteq {}^{\omega}\omega$ the properties $D(A)$ and $D_s(A)$, where $s \in {}^{<\omega}\omega$, are defined as follows:

- $D(A) \leftrightarrow (\forall f : {}^{<\omega}\omega \to \omega)(\exists x \in A)(\forall^\infty n \in \omega)\, x(n) \geq f(x \restriction n)$,

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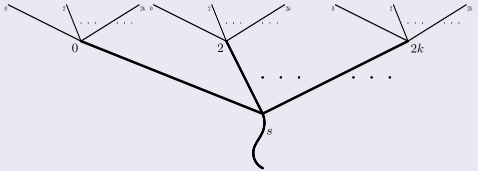
Example

$A_s = \{x \in [s] : (\forall n \geq |s|)\, x(n) \equiv 0 \mod 2\}$, where $s \in {}^{<\omega}\omega$.



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A tree $q \subseteq {}^{<\omega}\omega$ is said to be a *Laver tree*, if there is $s \in q$ (a *stem* of q) such that for every $t \in q$

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Every strongly dominating subset of the Baire space is dominating.

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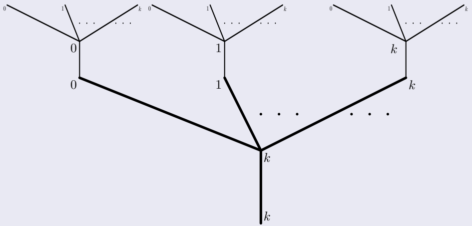
$B = \{x \in {}^{\omega}\omega : (\forall k \in \omega)\, x(2k+1) = x(2k)\}$.



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Theorem

The set $\mathcal{D}$ is σ-ideal on ${}^{\omega}\omega$ with base consisting of G_δ sets and cardinal characteristics as follows:

$$\mathrm{add}(\mathcal{D}) = \mathrm{cov}(\mathcal{D}) = \mathfrak{b}, \quad \mathrm{non}(\mathcal{D}) = \mathrm{cof}(\mathcal{D}) = \mathfrak{d}.$$

Moreover, ideal $\mathcal{D}$ is orthogonal to ideal $\mathcal{M}$ of meager sets and also to ideal $\mathcal{N}_\mu$ of sets of measure zero, for every finite atomless Borel measure μ on ${}^{\omega}\omega$.

Lemma

Let $A \subseteq {}^{\omega}\omega$ *and* $s \in {}^{<\omega}\omega \setminus \{\emptyset\}$ *be arbitrary. Then*

- $D(A) \leftrightarrow (\forall y \in {}^{\omega}\omega)(\exists x \in A)(\forall^{\infty} n \in \omega)\, x(n+1) \geq y(x(n))$.

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Corollary

If $D_s(A)$ *holds, then there is a Laver tree* $p \subseteq {}^{<\omega}\omega$ *with stem* s *such that for every* $x \in [p]$ *we have* $(\forall n \geq |s|)\, D_{x\restriction n}(A)$.

Analytic strongly dominating sets

Lemma

Let $A \subseteq {}^\omega\omega$ *and denote* $\Phi(A) = \{x \in {}^\omega\omega : (\forall^\infty k \in \omega)\, D_{x\restriction k}(A)\}$.
Then $A \setminus \Phi(A) \in \mathcal{D}$.

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Definition

For a family $\mathcal{A} \subseteq \mathcal{P}({}^{\omega}\omega)$ by induction on $\alpha < \omega_1$ we define

$$S_{\mathcal{A},0} = \{s \in {}^{<\omega}\omega : (\exists A \in \mathcal{A})\, D_s(A)\},$$

$$S_{\mathcal{A},\alpha} = \left\{s \in {}^{<\omega}\omega : (\exists^{\infty} k \in \omega)\, s^\frown\langle k\rangle \in \bigcup_{\beta<\alpha} S_{\mathcal{A},\beta}\right\},$$

$$\rho_{\mathcal{A}}(s) = \min\left\{\alpha \le \omega_1 : s \in S_{\mathcal{A},\alpha} \text{ or } \alpha = \omega_1\right\}, \text{ for } s \in {}^{<\omega}\omega.$$

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Remark

$\rho_{\mathcal{A}}(s) < \omega_1 \leftrightarrow (\exists^{\infty} k \in \omega)\, \rho_{\mathcal{A}}(s^\frown\langle k\rangle) < \rho_{\mathcal{A}}(s)$ for every $s \in {}^{<\omega}\omega$.

Lemma

If $\mathcal{A} \subseteq \mathcal{P}({}^{\omega}\omega)$ *and* $|\mathcal{A}| < \mathfrak{b}$*, then* $D_s\left(\bigcup \mathcal{A}\right)$ *holds if and only if* $\rho_{\mathcal{A}}(s) < \omega_1$*.*

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$$f(t) = \begin{cases} \min\{m \in \omega : (\forall k \geq m)\, \rho_{\mathcal{A}}(t^\frown\langle k\rangle) = \omega_1\}, & \text{if } \rho_{\mathcal{A}}(t) = \omega_1, \\ 0, & \text{otherwise.} \end{cases}$$

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4. Find $x \in \bigcup \mathcal{A} \cap \bigcup_{A \in \mathcal{A}} \Phi(A) \cap [s]$ such that $(\forall n \geq |s|)\, x(n) \geq f(x \restriction n)$.

Corollary

If $\mathcal{A} \subseteq \mathcal{P}({}^{\omega}\omega)$, $|\mathcal{A}| < \mathfrak{b}$ *and* $D_s(\bigcup \mathcal{A})$, *then there is a well-founded tree* $q \subset {}^{<\omega}\omega$ *with stem* s *such that*

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We call this tree an $\mathcal{A}$*-tree.*

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Definition

Let κ be an infinite cardinal. A subset of a Polish space X is κ-*Suslin*, if it is a continuous image of ${}^{\omega}\kappa$ (see [3]).

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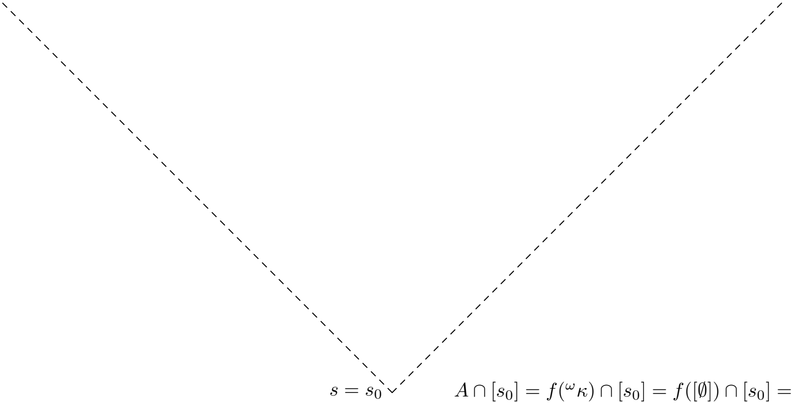
Definition

Let κ be an infinite cardinal. A subset of a Polish space X is *κ-Suslin*, if it is a continuous image of ${}^{\omega}\kappa$ (see [3]).

Theorem

Let $s \in {}^{<\omega}\omega$ be arbitrary. If a set $A \subseteq {}^{\omega}\omega$ is κ-Suslin for some $\kappa < \mathfrak{b}$, then the following conditions are equivalent:

1. *$D_s(A)$ holds.*
2. *There is a Laver tree $p \subseteq {}^{<\omega}\omega$ with stem s such that $[p] \subseteq A$.*



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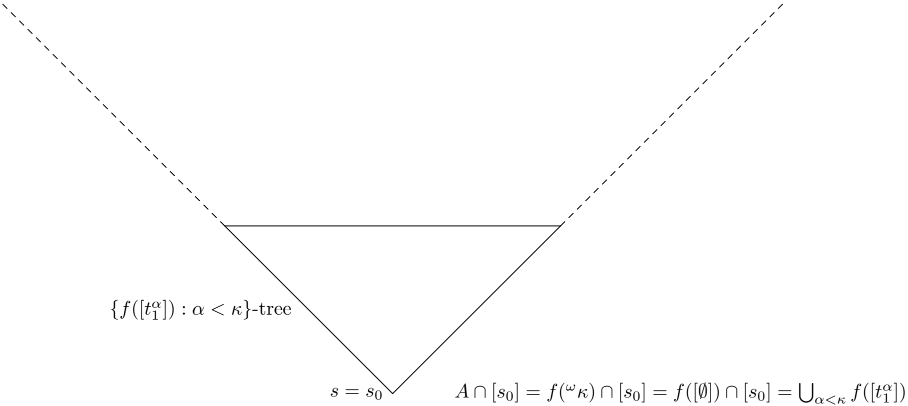
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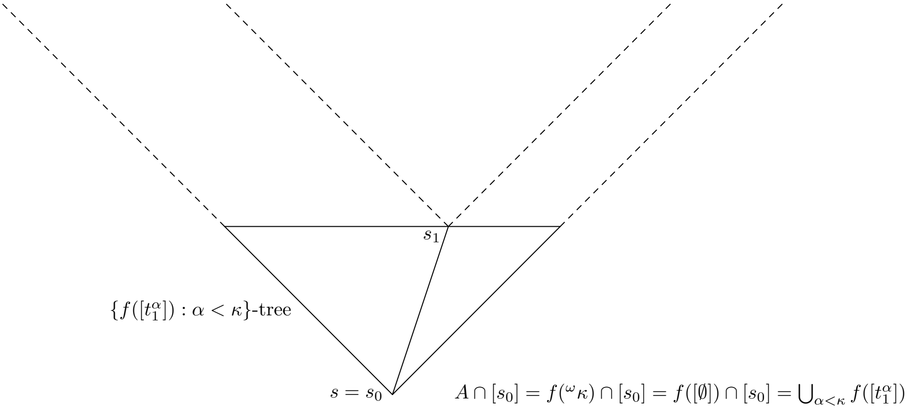
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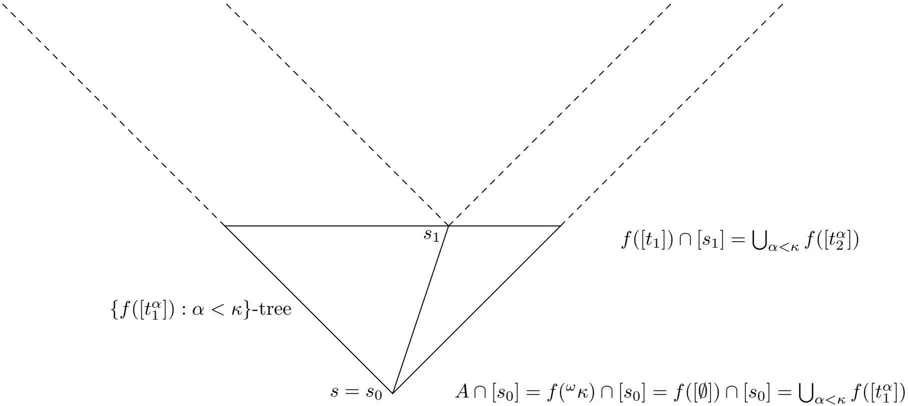
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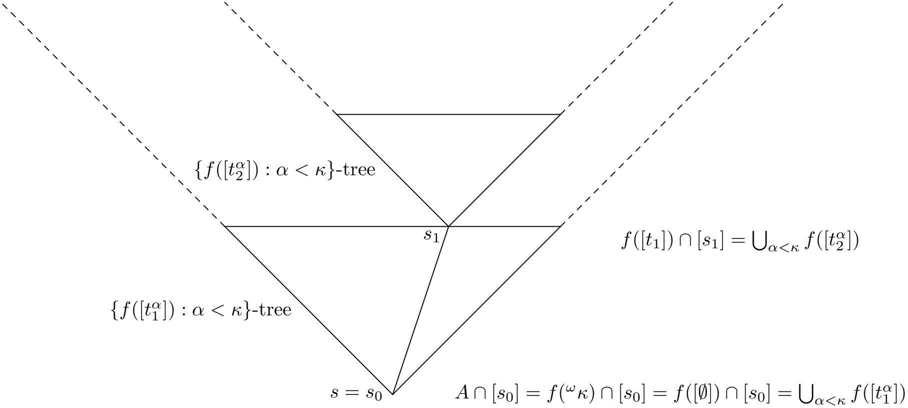
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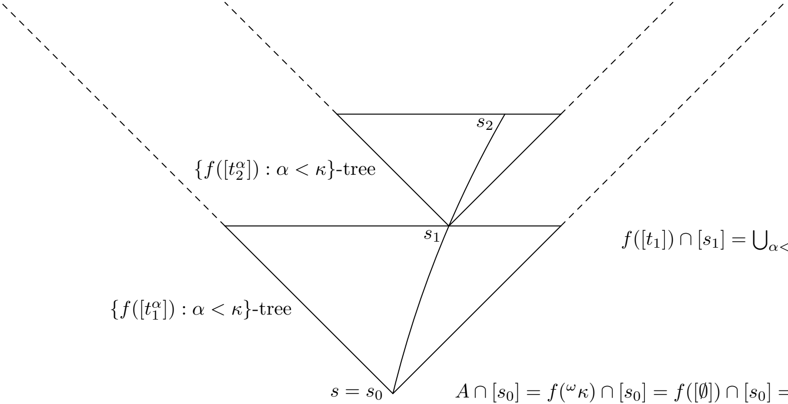

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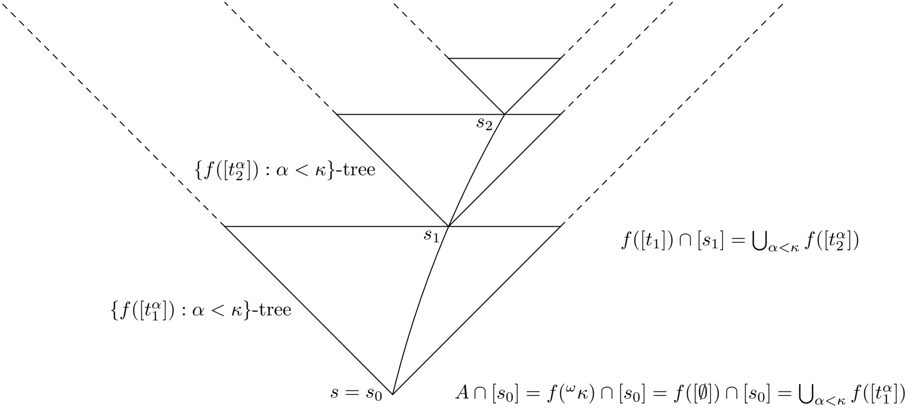
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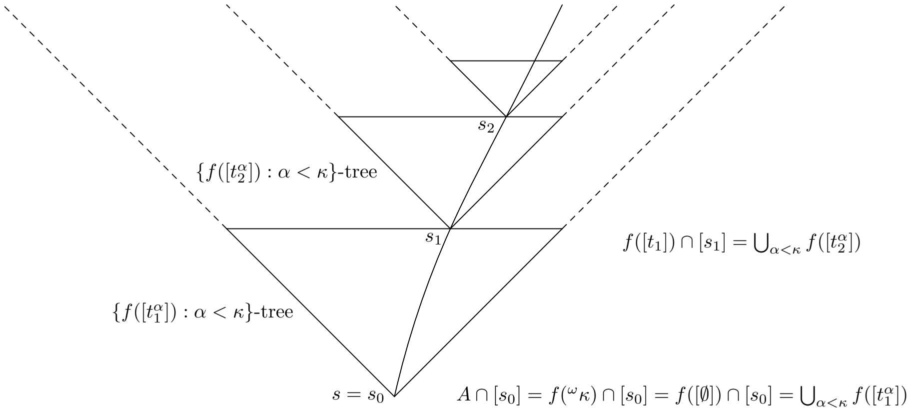
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Thank you for your attention.

Comparison of ideals $\mathcal{D}$ and l^0

Definition

Denote (see [1])

$$l^0 = \{X \subset {}^\omega\omega : (\forall q \in \mathbb{L})(\exists r \in \mathbb{L})\, r \subseteq q \text{ and } [r] \cap X = \emptyset\}.$$

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It is easy to see that $\mathcal{D} \subseteq l^0$ and $\mathcal{D} \cap \mathbf{\Sigma}^1_1 = l^0 \cap \mathbf{\Sigma}^1_1$.

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Theorem ([1])

$\mathfrak{t} \leq \mathrm{add}(l^0) \leq \mathrm{cov}(l^0) \leq \mathfrak{b}$ *and* $\mathrm{non}(l^0) = \mathfrak{c}$.

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2. if $t \in p(g)$ and $m \in \omega$, then $t^\frown\langle m\rangle \in p(g)$ iff $m \equiv g(t) \mod 2$.

Lemma

Let G be the set of functions $g : {}^{<\omega}\omega \to 2$ of cardinality less than $\mathfrak{c}$ and let $A = {}^{\omega}\omega \setminus \bigcup_{g\in G}[p(g)]$. Then $D_s(A)$ holds for every $s \in {}^{<\omega}\omega$.

Definition

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Theorem

$\mathcal{D} \neq l^0$.

Thank you for your attention, again.

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