Unique homogeneity, II

Jan van Mill

VU University Amsterdam and Technical University Delft

Hejnice 2013

Construction of a uniquely homogeneous space

Theorem (vM, 1983)

There is nontrivial separable metric UH Baire space.

There is a Boolean topological group G such that $G \approx \ell^2$. This means that $x + x = 0$ for every $x \in G$.
This space surfaces already in Halmos, Measure Theory. That it is homeomorphic to ℓ^2 was shown by Bessaga and Pełczyński (± 1970).
$\mathcal{M} = \{A \subseteq [0,1] : A$ measurable$\}$, $\mathcal{N} = \{A \in \mathcal{M} : \lambda(A) = 0\}$.
Consider $\mathcal{M}/\mathcal{N}$ with metric and group operation

$$d([A],[B]) = \lambda(A \triangle B), \qquad [A] + [B] = [A \triangle B].$$

Put Let $\mathcal{F}$ denote the collection of all functions f such that

1. $\mathrm{dom}(f)$ and $\mathrm{range}(f)$ are G_δ-subsets of G,
2. $f\colon \mathrm{dom}(f) \to \mathrm{range}(f)$ is a homeomorphism.

Let $\{f_\alpha : \alpha < \mathfrak{c}, \alpha \text{ even}\}$ enumerate $\mathcal{F}$, and let $\{K_\alpha : \alpha < \mathfrak{c}, \alpha \text{ odd}\}$ enumerate the collection of all Cantor subsets of G.

By transfinite induction on $\alpha < \mathfrak{c}$, we will construct subgroups H_α of G and subsets V_α of G such that the following conditions are satisfied:

1. if $\beta < \alpha$ then $H_\beta \subseteq H_\alpha$ and $V_\beta \subseteq V_\alpha$,
2. $H_\alpha \cap V_\alpha = \emptyset$, $|H_\alpha| \leq |\alpha| \cdot \omega$, $|V_\alpha| \leq |\alpha| \cdot \omega$,
3. if α is odd then $H_\alpha \cap K_\alpha \neq \emptyset$,
4. if α is even and

$$|\{x \in \operatorname{dom}(f_\alpha) : f_\alpha(x) \notin \langle\langle \bigcup_{\beta<\alpha} H_\beta \cup \{x\} \rangle\rangle\}| = \mathfrak{c}$$

then there exits $x \in \operatorname{dom}(f_\alpha) \cap (H_\alpha \setminus \bigcup_{\beta<\alpha} H_\beta)$ such that $f_\alpha(x) \in V_\alpha$.

Put $H^\alpha = \bigcup_{\beta<\alpha} H_\beta$, $V^\alpha = \bigcup_{\beta<\alpha} V_\beta$, and

$$S = \{x \in \mathrm{dom}(f_\alpha) : f_\alpha(x) \notin \langle\!\langle H^\alpha \cup \{x\} \rangle\!\rangle\}.$$

Observe that since G is Boolean, we have for every $x \in S$,

$$\langle\!\langle H^\alpha \cup \{x\} \rangle\!\rangle = H^\alpha \cup (x + H^\alpha).$$

Now assume first that α is even, that $|S| = \mathfrak{c}$, and pick $x \in S \setminus ((H^\alpha + V^\alpha) \cup H^\alpha)$. It is clear that such an x exists by cardinality considerations. Now put

$$H_\alpha = \langle\!\langle H^\alpha \cup \{x\} \rangle\!\rangle = H^\alpha \cup (x + H_\alpha), \qquad V_\alpha = V^\alpha \cup \{f_\alpha(x)\}.$$

Then $H_\alpha \cap V_\alpha = \emptyset$. The case that α is odd can be treated analogously since every Cantor set has size $\mathfrak{c}$.
Put $H = \bigcup_{\alpha<\mathfrak{c}} H_\alpha$. We claim that H is UH.

H is a Baire space since it intersects all Cantor subsets of the Polish space G (observe that it hits every dense G_δ-subset of H since such a set contains a Cantor set, hence H is of the second category in itself and hence a Baire space being a second countable topological group).
Let $f\colon H \to H$ be a homeomorphism. By Lavrentieff, there exist G_δ-subsets A and B in G such that f can be extended to a homeomorphism $\bar{f}\colon A \to B$. Pick α such that $\bar{f} = f_\alpha$.

CASE 1: $|\{x \in A : f_\alpha(x) \notin \langle\!\langle \bigcup_{\beta<\alpha} H_\beta \cup \{x\} \rangle\!\rangle = \mathfrak{c}\}$.

Then at stage α we picked $x \in H_\alpha$ such that $f_\alpha(x) \in V_\alpha$. Hence there exists $x \in H$ such that $f_\alpha(x) \notin H$. But f_α extends f, hence $f_\alpha(x) = f(x) \in H$, which is a contradiction.

CASE 2: If $T = \{x \in A : f_\alpha(x) \notin \langle\!\langle H^\alpha \cup \{x\} \rangle\!\rangle\}$, where $H^\alpha = \bigcup_{\beta<\alpha} H_\beta$, then $|T| < \mathfrak{c}$.

For $h \in H^\alpha$, put $E_h = \{x \in A : f_\alpha(x) = x + h\}$. Then E_h is a closed subset of A, $E_h \cap T = \emptyset$, and $E_h \cap E_{h'} = \emptyset$ if $h \neq h'$.
Put $F_h = E_h \setminus f_\alpha^{-1}(H^\alpha)$.

CLAIM: At most one F_h is nonempty.

Assume that there are distinct $s, t \in H^\alpha$ such that both F_s and F_t are nonempty, say $x \in F_s$ and $y \in F_t$.
Observe that $G \setminus A$ is countable, since otherwise it would contain a Cantor set which would intersect H by construction, which is impossible since A contains H.
Hence

$$|f_\alpha^{-1}(H^\alpha) \cup (G \setminus A) \cup T| < \mathfrak{c}$$

and since $x, y \notin E = f_\alpha^{-1}(H^\alpha) \cup (G \setminus A) \cup T$, there is an arc J in G that connects x and y and misses E. Observe that

$$J \subseteq \bigcup_{h \in H^\alpha} F_h \subseteq \bigcup_{h \in H^\alpha} E_h.$$

Put $K = \{h \in H^\alpha : F_h \cap J \neq \emptyset\}$. By assumption, $|K| \geq 2$. Hence $|K| > \omega$ by the Sierpinski Theorem. Since $|K| < \mathfrak{c}$, we have a contradiction in the presence of the CH.
K is not complete being uncountable and of size less than $\mathfrak{c}$, and hence not closed in G. Pick $k_n \in K$ such that $k_n \to h \notin K$. For every n pick $x_n \in J \cap F_{k_n}$. We may assume without loss of generality that $x_n \to x$. Then

$$f_\alpha(x) = \lim_{n\to\infty} f_\alpha(x_n) = \lim_{n\to\infty} x_n + k_k = x + k.$$

Since $x \in J$, there exists $h \in K$ such that $x \in F_h \subseteq E_h$. Hence $f_\alpha(x) = x + h$ so that $k = h \in K$, contradiction.

CLAIM: At least one of the collection $\{F_h : h \in H^\alpha\}$ is nonempty.

If not, then $f_\alpha(H \setminus T) \subseteq H^\alpha$, which is a contradiction since f_α is injective.

Hence there is a unique $h \in H^\alpha$ such that $F_h \neq \emptyset$. Now consider E_h. The complement of A is countable, as we observed above. The set E_h is closed, hence if it would be a proper subset of A its complement would have cardinality $\mathfrak{c}$. But it has size less than $\mathfrak{c}$. This implies that $E_h = A$, hence $f_\alpha(x) = x + h$ for every $x \in A$.

This implies that every homeomorphism of H is a translation of the form $x \mapsto x + h$, hence H is uniquely homogeneous.

The construction can be improved so that H has the following property: every continuous function $f\colon H \to H$ is either constant, or a translation.

The weight of H is ω.

Question

Are there UH spaces of arbitrarily large weight?

Arhangelskii and vM (2012) proved that there is a family $\{H_\alpha : \alpha < 2^{\mathfrak{c}}\}$ of such groups such that if $\alpha \neq \beta$, then every continuous function $f \colon H_\beta \to H_\alpha$ is constant. This implies that the product $\prod_{\alpha<2^{\mathfrak{c}}} H_\alpha$ is UH and has weight $2^{\mathfrak{c}}$.
But this is cheating, it is not a new construction.

A space X is called *2-flexible* if $\forall a, b \in X$, $\forall O_b$, $\exists O_a$, $\forall z \in O_a$, $\exists h \in H(X)$, $h(a) = z$ and $h(b) \in O_b$.
A space X is called *skew 2-flexible* if $\forall a, b \in X$, $\forall O_b$, $\exists O_a$, $\forall z \in O_a$, $\exists h \in H(X)$, $h(a) = z$ and $b \in g(O_b)$.

Theorem

If X is locally compact, separable metric, homogeneous, then X is both 2-flexible and skew 2-flexible.

Application of the Effros Theorem

Example

1. There is a homogeneous Polish space which is skew 2-flexible but not 2-flexible. *Hence Effros does not work for Polish spaces.*
2. There is a UH 2-flexible space that is not skew 2-flexible. *Open whether such a space can be Polish.*

Theorem (Arhangelskii and vM)

Let X be UH. TFAE

1. *X is 2-flexible.*
2. *X is Abelian. (for all $f, g \in H(X)$ we have $f \circ g = g \circ f$.)*

Theorem (Arhangelskii and vM)

Let X be UH. TFAE

1. *X is skew 2-flexible.*
2. *X is Boolean. (for all $f \in H(X)$ we have $f \circ f = id_X$.)*

Hence for a UH space, skew 2-flexibility implies 2-flexibility. The converse is not true.