# Unique homogeneity, I

Jan van Mill

VU University Amsterdam and Technical University Delft

Hejnice 2013

A topological space $X$ is called *(topologically) homogeneous* if for all $x, y \in X$ there is a homeomorphism $f\colon X \to X$ such that $f(x) = y$.

A topological space $X$ is called *uniquely homogeneous* (abbreviated UH) if for all $x, y \in X$ there is a *unique* homeomorphism $f\colon X \to X$ such that $f(x) = y$.

$X = \{0\}$ and $X = \{0, 1\}$ are UH.

Burgess, 1955. *Is there a UH metric continuum with more than one point?*

Ungar, 1975, NO for finite dimensional continua. Barit and Renaud, 1978, NO.

**Theorem**

*If $X$ is locally compact, metric and UH, then $|X| \leq 2$.*

Eric van Douwen, 1979, Baton Rouge.

If $X$ is UH, and $x, y \in X$, let $f_y^x$ be the unique homeomorphism taking $x$ onto $y$.
Fix an element $e \in X$. We denote $f_x^e$ by $f_x$ for brevity. Define a binary operation '$\cdot$' on $X$ by

$$x \cdot y = f_x(y).$$

This is a group operation.
$x^{-1} = f_x^{-1}(e)$. $\qquad x \cdot x^{-1} = f_x(x^{-1}) = f_x(f_x^{-1}(e)) = e.$

$(f_x f_y)(e) = f_x(f_y(e)) = f_x(y) = x \cdot y = f_{x \cdot y}(e)$. $\qquad f_{x \cdot y} = f_x \circ f_y$.

$$x(yz) = xf_y(z) = f_x(f_y(z)) = (f_x f_y)(z)$$
$$(xy)z = f_{xy}(z) = (f_x f_y)(z)$$

Etc.

He tried to prove that this operation gives $X$ the structure of a topological group.

Not such a bad idea since all left-translations of $X$ are homeomorhisms.

$x \mapsto a \cdot x$ is equal to $f_a$

*So every UH space has the structure of a left-topological group*

### Theorem

1. *(vM, 1983) There is a uniquely homogeneous separable metric Baire space of size $\mathfrak{c}$ which is a topological group.*
2. *(vM, 1984) There is a uniquely homogeneous separable metric Baire space of size $\mathfrak{c}$ which does not admit the structure of a topological group.*

After these results, nothing happened with unique homogeneity, until 2012 by the work of Arhangelskii and vM

### Question

1. Does there exist a Polish UH space? *It cannot be locally compact*
2. Is there a compact uniquely homogeneous space? *It cannot be metrizable and it cannot be a topological group (W. Rudin)*
3. Do there exist uniquely homogeneous spaces of arbitrarily large weight?

### Theorem

1. *Every UH space of size greater than 2 is connected*
2. *No ordered space of size greater than 2 is UH*

Proof of the Barit and Renaud Theorem that no locally compact space of size greater than 2 is UH.

We only do the compact case.

### Theorem

*If $G$ is an analytic group acting transitively on a second category separable metric space $X$, then for every $x \in X$, the evaluation mapping*

$$G \to X \qquad g \mapsto gx$$

*is a continuous and open surjection.*

This is the so-called *Effros Theorem*. It has some very interesting consequences.

*Banach Open Mapping Theorem*, in a strengthened form.

**Theorem**

*If $G$ and $H$ are topological groups, $G$ analytic and $H$ Polish, and $\varphi\colon G \to H$ is a continuous surjective homomorphism, then $\varphi$ is open.*

Let $e$ denote the neutral element of $H$.

$$G \times H \to H \qquad (g, h) \mapsto \varphi(g) \cdot h.$$

Evaluate this action at the point $e$:

$$g \mapsto \varphi(g) \cdot e = \varphi(g).$$

Hence this evaluation is $\varphi$, hence $\varphi$ is open.

*Ungar's Theorem from 1975.*

For a space $X$ we let $H(X)$ denote its group of homeomorphisms.

**Theorem**

*If $X$ is a homogeneous compact metric space, then $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $\forall\, x, y \in X\ d(x, y) < \delta\ \exists h \in H(X)\ h(x) = y$ and $h$ moves no point more than $\varepsilon$.*

Fix $\varepsilon > 0$, and let $V$ denote the open neighborhood of the neutral element $e$ of $H(X)$ consisting of those homeomorphisms that move every point less than $^1\!/_2\varepsilon$. Then $Vx$ is open in $X$ for every $x \in X$ by Effros. Now let $\delta$ be a Lebesgue number for the open cover $\{Vx : x \in X\}$ of $X$.

*Proof of the Barit and Renaud Theorem from 1978*

Fix $x \in X$. The evaluation at $x$ $\gamma_x : h \mapsto hx$ is an open, continuous surjection by Effros. It is also injective by unique homogeneity:

$$hx = \gamma_x(h) = \gamma_x(g) = gx$$

gives us that $(g^{-1}h)x = x$, i.e., $g^{-1}h = e$. Hence $H(X)$ and $X$ are homeomorphic. This means that $X$ is a compact topological group. Consider the homeomorphism $x \mapsto x^{-1}$. This function fixes $e$, hence is the identity. So $x = x^{-1}$ for every $x \in X$. Hence $X$ is a Boolean topological group, hence Abelian. Structure theorems of compact Abelian groups then give that $X$ is homeomorphic to $\{0,1\}^\kappa$ for some $\kappa \leq \omega$. But this is not uniquely homogeneous for $\kappa \geq 2$.