# CONVERGENCE AND CHARACTER SPECTRA OF COMPACT SPACES

István Juhász
juhasz@renyi.hu

Alfréd Rényi Institute of Mathematics

Hejnice,
February, 2010

# Overview

- Basic definitions
- Hušek's problem
- Inclusion in spectra
- Omission by spectra
- A problem on the $G_\delta$-topology

# convergence spectrum

$A \to p$ if, for every neighbourhood $U$ of $p$, $|A \setminus U| < |A|$

$$cS(p,X) = \{|A| : A \subset X \text{ and } A \to p\}$$

is the convergence spectrum of $p$ in $X$

$$cS(X) = \cup\{cS(x,X) : x \in X\}$$

is the convergence spectrum of $X$

$\chi(p,X) = \psi(p,X) = \kappa \geq \omega \Rightarrow$ there is a 1-1 sequence $\langle x_\alpha : \alpha < \kappa \rangle$ with $x_\alpha \to p$; hence $\kappa, \mathrm{cf}(\kappa) \in cS(p,X)$

In a compact $T_2$ space $X$, $\chi(p,X) = \psi(p,X)$ for all points $p \in X$

$$\chi S(p,X) = \{\chi(p,Y) : p \text{ is non-isolated in } Y \subset X\}$$

is the character spectrum of $p$ in $X$

$$\chi S(X) = \cup\{\chi S(x,X) : x \in X\}$$

is the character spectrum of $X$.

If $X$ is compact $T_2$ then

$$\chi(p,Y) = \chi(p,\overline{Y})$$

for any $p \in Y \subset X$, so we may restrict to closed (i.e. compact) subspaces. This also implies:

For $X$ compact $T_2$,

$$\chi S(p,X) \subset cS(p,X) \text{ and } \kappa \in \chi S(p,X) \Rightarrow \text{cf}(\kappa) \in cS(p,X)$$

# Hušek's Problem

From here on, unless otherwise stated, space (usually denoted $X$) is compactum $\equiv$ infinite compact $T_2$ space

Note: $\omega \in cS(X) \Leftrightarrow \omega \in \chi S(X)$ and $\min cS(X) \leq \min \chi S(X) \leq 2^\omega$

Alexandrov-Urysohn (1920's) : Is $\omega \in cS(X)$ ?

NO! Tychonov (1935), Čech, (1937) : $\omega \notin cS(\beta\omega)$

M. Hušek (1970's) : Is $\min cS(X) \leq \omega_1$ ?

A. Dow (1989) : $V^{\mathbb{C}_\kappa} \models$ YES , if $V \models$ CH

I. J. (1993) : $V^{\mathbb{C}_{\omega_1}} \models \min \chi S(X) \leq \omega_1$ , for any $V$

I conjecture that ZFC $\vdash \min \chi S(X) \leq \omega_1$ , but don't even know if

$$\text{ZFC} \vdash \chi S(X) \cap \text{REG} \neq \emptyset \ !?$$

# free sequences

## DEFINITION.

$\{x_\alpha : \alpha < \varrho\}$ is free in $X$ if, for all $\alpha < \varrho$,

$$\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset$$

## THEOREM. (J – Szentmiklóssy, 1991)

If there is a free sequence of length $\varrho = \mathrm{cf}(\varrho) > \omega$ in $X$ then there is one converging to some $p \in X$. Moreover, then

$$\chi(p, \overline{\{x_\alpha : \alpha < \varrho\}} = \varrho\,.$$

Arhangel'skii : $X$ is countably tight iff it has no uncountable free sequences. Hence Hušek's problem is about countably tight compacta.

My original conjecture (true in $V^{\mathbb{C}_{\omega_1}}$) : Any countably tight compactum has a point of character $\leq \omega_1$ (maybe isolated!).

# main lemma for inclusion

Non-attributed results below are joint with W. Weiss

$$\widehat{F}(X) = \min\{\kappa : \neg\exists \text{ free sequence of length } \kappa \text{ in } X\}$$

**MAIN LEMMA.**

Let $X$ be a $T_3$ space with $\widehat{F}(X) \le \varrho \le \mathrm{cf}(\mu)$, moreover $p \in X$ with $\psi(p, X) \ge \mu$. Then either
(i) there is a discrete $D \in [X]^{<\varrho}$ with $p \in \overline{D}$ and $\psi(p, \overline{D}) \ge \mu$, or
(ii) there is a discrete $D \in [X]^{\varrho}$ such that $D \to p$.

$$\widehat{t}(X) = \min\left\{\kappa : \forall A \subset X \left(\overline{A} = \cup\{\overline{B} : B \in [A]^{<\kappa}\}\right)\right\}$$

Arhangel'skii : $\widehat{t}(X) \le \widehat{F}(X) \le \widehat{t}(X)^+$ and if $\widehat{t}(X)$ is regular then $\widehat{t}(X) = \widehat{F}(X)$. In particular, $X$ is countably tight iff

$$\widehat{t}(X) = \widehat{F}(X) = \omega_1$$

**THEOREM 1.**

If $\chi(p, X) > \lambda = \lambda^{<\widehat{t}(X)}$ then $\lambda \in \chi S(p, X)$. So, if $X$ is countably tight and $\chi(p, X) > \lambda = \lambda^{\omega}$ then $\lambda \in \chi S(p, X)$.

COROLLARY. $\chi(X) > \mathbf{c}$ implies $\omega_1 \in \chi S(X)$ or $\{\mathbf{c}, \mathbf{c}^+\} \subset \chi S(X)$. So, if $\chi(X) > \omega$ then $\chi S(X) \cap [\omega_1, \mathbf{c}] \neq \emptyset$.

COROLLARY. If $\kappa$ is strong limit and $|X| \geq \kappa$ then

$$\sup\left(\kappa \cap \chi S(X)\right) = \kappa .$$

NOTATION. $dcS(p,X) = \{|D| : D \subset X \text{ is discrete and } D \to p\}$

$$dcS(X) = \cup\{dcS(x,X) : x \in X\}$$

**THEOREM 2.**

$\widehat{F}(X) \le \lambda = \mathrm{cf}(\lambda)$ and $\chi(p,X) \ge \sum\{(2^\kappa)^+ : \kappa < \lambda\} \Rightarrow \lambda \in dcS(p,X)$.

COROLLARY. If $\chi(X) > 2^\kappa$ then $\kappa^+ \in dcS(X)$.

So, $\chi(X) > \mathbf{c} \Rightarrow \omega_1 \in dcS(X)$.

# omitting $\omega$

$S$ omits $\kappa$ if $\kappa \notin S$ but there is a $\lambda \in S$ with $\lambda > \kappa$.

Tychonov (1935), Čech, (1937) : $\omega \notin cS(\beta\omega)\big( \Leftrightarrow \omega \notin \chi S(\beta\omega)\big)$;
under CH, $\chi S(\beta\omega) = \{\omega_1\}$.

Fedorchuk (1977) : $\mathbf{s} = \omega_1$ implies $\exists X$ with $\chi S(X) = \{\omega_1\}$ ;
if $2^{\omega_1} < \aleph_{\omega_1}$ then $cS(X) = \{\omega_1\}$ as well. But

$$\{\lambda < 2^{\omega_1} : \mathrm{cf}(\lambda) = \omega_1\} \subset cS(X).$$

If $\mathbf{p} > \omega_1$ then $\chi S(X) \neq \{\omega_1\}$ for all $X$.

# omitting uncountable cardinals 1.

The cardinality spectrum $S(X)$ of any top. space $Y$ is the set of cardinalities of all infinite closed subspaces of $Y$.

## Lemma

Let $Y$ be a locally compact $T_2$ space which is also locally $\mu$, and let $X = Y \cup \{p\}$ be the one-point compactification of $Y$. If $\mu < \kappa < |Y|$ and $\kappa \notin S(Y)$ then $\kappa \notin \chi S(X)$, while $|Y| = \chi(p, X)$.

## $\Phi(\kappa)$

There are $T \in [\mathbb{R}]^\kappa$ and $\mathcal{A} \subset [T]^\omega$ with $|\mathcal{A}| = \kappa$ such that (i) for every $A \in \mathcal{A}$ we have $|T \cap \overline{A}| = \kappa$ and (ii) for every $B \in [T]^{\omega_1}$ there is $A \in \mathcal{A}$ with $A \subset B$.

## Theorem

$\Phi(\kappa) \Rightarrow \exists$ locally countable and locally compact $T_2$ space $Y$ with $S(Y) = \{\omega, \kappa\}$, hence an $X$ with $\chi S(X) = \{\omega, \kappa\}$.

# omitting uncountable cardinals 2.

$\Phi(\mathbf{c})$ is (trivially) true.

COROLLARY. (Hušek, 1981) $\exists X$ s.t. $\chi S(X) = \{\omega, \mathbf{c}\}$.

## Lemma

If $\kappa \le \mathbf{c}$ with $\mathrm{cf}(\kappa) \ne \omega_1$ and $\langle [\kappa]^{\omega_1}, \subset \rangle$ has a dense subfamily of size $\kappa$ then $\Phi(\kappa)$ holds.

## Proposition

Let $\lambda$ be singular of countable cofinality s.t. $\mu^{\omega_1} < \lambda$ whenever $\mu < \lambda$. For every CCC partial order $\mathbb{P}$ with $|\mathbb{P}| = \lambda$, $\langle [\lambda]^{\omega_1}, \subset \rangle$ has a dense subfamily of size $\lambda$ in $V^{\mathbb{P}}$. (A. Miller, for $\mathbb{P} = \mathbb{C}_\lambda$ )

## Corollary

If $V \models GCH$ then, for any $\kappa > \omega$, $V^{\mathbb{C}_\kappa} \models \Phi(\kappa)$.

# omitting uncountable cardinals 3.

**Theorem**

Suppose $V \models GCH$ and $\lambda > \omega$ is a cardinal in $V$. Then, in $V^{\mathbb{C}_\lambda}$, for every $\kappa \le \mathbf{c}$ there is a locally countable and locally compact $T_2$ space $Y$ with $S(Y) = \{\omega, \kappa\}$, hence there is a compactum $X$ with character spectrum $\chi S(X) = \{\omega, \kappa\}$.

Proof: $V^{\mathbb{C}_\lambda} = \left(V^{\mathbb{C}_\kappa}\right)^{\mathbb{C}_{\lambda\setminus\kappa}}$ and the properties of $Y$ are preserved.

**Corollary**

In $V^{\mathbb{C}_\lambda}$, for every countable set $A$ of cardinals with $\omega \in A \subset [\omega, \mathbf{c}]$ there is $X$ s.t. $\chi S(X) = A$.

**Theorem (L. Soukup)**

It is consistent with $\mathbf{c}$ big that $\Phi(\kappa)$ holds for all $\kappa \le \mathbf{c}$.

Each example $X$ so far is the one-point compactification of a locally countable (loc. cpt) space, hence satisfies

$$cS(X) = [\omega, |X|] \, .$$

**Theorem (J-Koszmider-Soukup, 2009)**

Consistently, there is $X$ s.t.

$$\chi S(X) = cS(X) = \{\omega, \omega_2\} \, .$$

This is the only known example whose convergence spectrum is not convex on REG!

# $G_\delta$ – topology

**FACT.**

Any crowded $X$ has a crowded, hence non-discrete countable subspace.

**PROBLEM.**

If $\chi(p, X) > \omega$ for all $p \in X$, does $X_\delta$ have a non-discrete subspace of size $\omega_1$?

YES, if $\omega_1 \in cS(X)$, hence YES if $X$ is not countably tight.

YES for all $X$, if my old conjecture holds.