Moron Maps and subspaces of N* extending PFA

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Question 1 could such a point be selective?

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although, P_3 does add a 4-point

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I was intrigued by the quote: P_2 adds an automorphism "while doing as little else as possible".

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Questions galore: e.g. force with P_2, is every 2-point RK-equivalent to the generic x?

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We will use the Shelah-Steprans technique for producing new elements of $\mathbb{P}$ (representing one of the posets described above). It uses the CH trick.

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or, if $\dot{h}$ is a code for a dense G_δ in $\mathbb{R}$, then there can be an r such that $f \Vdash r \in \lceil \dot{h} \rceil$

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This shows that MA $+\neg$CH does not imply all automorphisms are trivial.

$\Vdash \mathbb{P}(\mathcal{F})$ is ω^ω-bounding and preserves category

It's a simple matter to force an $I \subset \mathbb{N}$ so that $\limsup |\iota^n \setminus I|$ is infinite, and $I \supset^* dom(f)$ for all $f \in \mathcal{F}$. Thus, if $\mathcal{F}_G \subset \mathcal{F}$ is $\mathbb{P}(\mathcal{F})$-generic, $p_{\mathcal{F}} = \bigcup\{f \restriction I : f \in \mathcal{F}_G\}$ is a special member of $\mathbb{P}$.

How special? For any prescribed family $\mathcal{D}$ of $< \aleph_2$ dense open subsets of $\mathbb{P}$, $p_{\mathcal{F}} \in \bigcap \mathcal{D}$ (and whatever else we can make it force)

Thus $\mathbb{P}$ is $\aleph_2$-distributive (preserves MA(ω_1) and cardinals).

This shows that MA $+\neg$CH does not imply all automorphisms are trivial.

Question 2 Does MA $+\neg$CH imply $\mathcal{P}(\mathbb{N})$ is not $\mathfrak{c}$-universal?

sample preservation of PFA theorems

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Fundamental local Lemma for $\mathbb{P}(\mathcal{F})$

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Apply to gaps: obviously Case 1 implies that $\dot{h}^{-1}(0)$ does not split the gap. But similarly with Case 2 because ...

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Thus! after forcing with $\mathbb{P}(\mathcal{F})$, we then select proper poset Q to freeze the gap, then force with the nice σ-centered poset to get $p_{\mathcal{F}}$ which forces that $\{c_\alpha : \alpha \in \omega_1\}$ and $\{d_\beta : \beta \in \lambda\}$ is a gap (and so $\dot{d}_\lambda$ can't exist).

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Corollary: Since we fail, the ideal of sets on which $\Phi \restriction V$ is σ-Borel is ccc over fin holds in the extension by $\mathbb{P}(\mathcal{F})$,

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Pulling this back and up to the generic extension by $\mathbb{P}$, this describes a dense P_{ω_2}-ideal, $\mathfrak{J}$, contained in $triv(\Phi)$.

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But still a lot can happen in the large complement. Remember we have the generic ultrafilter x, which induces an ultrafilter y by the finite-to-one map $\psi([n_k, n_{k+1})) = k$, and so the behavior of Φ on the large set $y - \lim\{[n_k, n_{k+1}) : k \in \omega\}$ is still unknown, and this is where we expect all the action to be.

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Theorem: There is a $\mathbb{P}(\mathcal{F})$-name $\dot{h} \in \mathbb{N}^{\mathbb{N}}$ which is forced to mod finite contain h_I for all $I \in \mathit{triv}(\Phi)$.

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The proof follows our pattern: We have our dense P_{ω_2}-ideal of functions. If forcing with $\mathbb{P}(\mathcal{F})$ adds no extension, then there is a proper poset freezing this fact. Meeting ω_1 many dense sets pulls back to an $\aleph_1$-sized subfamily of our dense P_{ω_2}-ideal which can not have a common extension – contradicting that it's a P_{ω_2}-ideal.

making sense of $\dot{h}$ from local Lemma

For each k we are still assuming there is a single m_k such $S_k = \iota^{m_k} \setminus dom(f) \subset [n_k, n_{k+1})$ is non-empty. and that the fundamental lemma ensured that

the values of $\dot{h} \restriction [n_k, n_{k+1})$ are just determined by functions $s : S_k \mapsto S_k$

so we can also assume that $f \Vdash \dot{h}([0, n_k]) \subset n_{k+1}$ and that for each $j < n_k$ and each $s : n_{k+1} \mapsto n_{k+1}$, such that $g = f \sqcup s < f$, if there is no $i \in a_g \cap n_{k+1}$ such that $\dot{h}(i) = j$, then this is true for all $\bar{f} < f \sqcup s$.

We can now complete the 2-to-1 image problem: obtain $A_1 \oplus_{x_2}^{x_1} B_2 \not\approx \mathbb{N}^*$ with propellers $A_i \oplus_{x_i} B_i$

a 2-to-1 image which is not $\mathbb{N}^*$

For this we force with $\mathbb{P} = P_{2,2}$ and assume that we have $A_1 \oplus^{x_1}_{x_2} B_2 \quad \approx_\varphi \mathbb{N}^*$. This implies the existence of a pair of homomorphisms, which we combine and call Φ where $\Phi_1(X)^* = \varphi^{-1}(X^* \cap A_1)$ and $\Phi_2(X)^* = \varphi^{-1}(X^* \cap B_2)$.

our $\dot{h}$ will induce Φ on all X such that $X^* \subset A_1 \cup B_2$. Let $\dot{z}$ denote the $\mathbb{P}$-name of the ultrafilter on $\mathbb{N}$ ($\varphi(z) = \{x_1, x_2\}$) to which each of x_1 and x_2 are sent (i.e. $\Phi(X) \notin \dot{z}$ for all X with $X^* \subset A_1 \cup B_2$). It follows easily then that for all f and all $X \in x_1 \cup x_2$,
$\{j : (\exists g < f, i \in X) \;\; i \in a^1_g \cup b^2_g$ and $g \Vdash \dot{h}(i) = j\}$ is in $\dot{z}$

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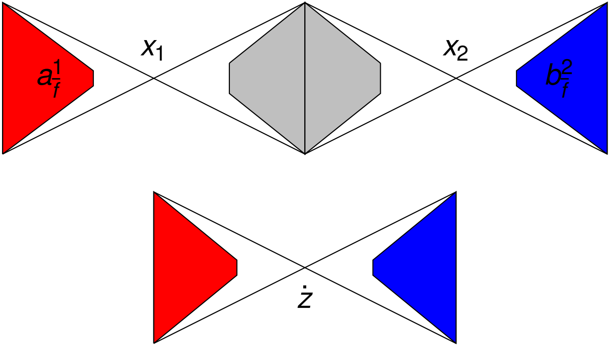
We can strengthen f and have $\bigcup_k [n_{3k+1}, n_{3k+3}) \subset dom(f)$.

Recall $E = \bigcup_j \iota^{2j} \in x_1 \setminus x_2$: choose any $\bar{f} < f$ such that $\bar{f}$ force a value on $\Phi(a^1_{\bar{f}} \cup b^2_{\bar{f}})$ (not in z).



Let $Y_1 = \{j : (\exists g < \bar{f})\ (\exists i \in a^1_g)\ g \Vdash \dot{h}(i) = j\}$ and $Y_2 = \{j : (\exists g < \bar{f})\ (\exists i \in b^2_g)\ g \Vdash \dot{h}(i) = j\}$ (both are in $\dot{z}$)

fix any $j \in Y_1 \cap Y_2 \setminus \Phi(a_f^1 \cup b_f^2)$, and $g_1, g_2 < f$ i_1, i_2 witnessing $j \in Y_1 \cap Y_2$. Let $j \in [n_k, n_{k+1})$ and (wlog) $\iota^{m_k} \subset \mathbb{N} \setminus E$.

By our construction,
since there is some i with $i \in a_g^1$ such that $g = g_1 \cup f \Vdash \dot{h}(i) = j$,

there must be an $i \in [n_k, n_{k+2}) \cap a_f^1$ such that $g_1 \cup f \Vdash \dot{h}(i) = j$.

However this contradicts that $g_1 \Vdash j \notin \Phi(a_f^1 \cup b_f^2)$,

and that $\bar{f}$ forces $\dot{h} \supset^* h_{a_f^1}$.

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there must be an $i \in [n_k, n_{k+2}) \cap a^1_f$ such that $g_1 \cup f \Vdash \dot{h}(i) = j$.

However this contradicts that $g_1 \Vdash j \notin \Phi(a^1_f \cup b^2_f)$,

and that $\bar{f}$ forces $\dot{h} \supset^* h_{a^1_f}$.

one of the things that is going on is that things about Φ are forced by $\mathbb{P}$, while things about $\dot{h}$ are forced by $\mathbb{P}(\mathcal{F})$

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Key Lemma The condition f and sequence $\{n_k\} \nearrow$ can be chosen so that there is a partial function $\psi : \mathbb{N} \mapsto \mathbb{N} \setminus \mathit{dom}(f)$ so that for all $i \notin \mathit{dom}(f)$, $\psi^{-1}(i) \subset [n_k, n_{k+1})$ for some k, and for all $g < f$, g forces a value on $\dot{h} \restriction \psi^{-1}(i)$ iff $f \cup \{(i, g(i))\}$ forces this value.

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Let L be the domain of ψ. It follows that if Φ is not trivial, then $L \notin \mathit{triv}(\Phi)$ (but we skip).

so all automorphisms are trivial

Choose just *any* total function g extending f; but for definiteness assume that $g(i) = 0$ for all $i \notin dom(f)$.
This defines a ground model function h as an interpretation of $\dot{h}$, i.e. $h(\ell) = j$ if $\psi(\ell) = i$ and $f \cup \{(i, 0)\} \Vdash \dot{h}(\ell) = j$. We know that this function h does not induce Φ, so it is easy to show that there is an infinite set $Y \subset L$ such that $h[Y] \cap F(Y)$ is empty.

It's simple enough to now shrink Y and arrange that $K = \{k : Y \cap [n_k, n_{k+1}) \neq \emptyset\}$ and $J = \bigcup_{k \in K} [n_k, n_{k+1})$, are such that $f \cup g \restriction J$ is a condition. This condition forces that $\dot{h}$ does not extend h_J despite the fact that $J \in \mathfrak{J} \subset triv(\Phi)$.

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We choose $f_j < f_{j-1}$ by a length 2^{j+1} induction.

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For a condition $g \in \mathbb{P}$ and function $\psi \in 2^{j+1}$, define

g^ψ by redefining g at all values in $\{i_\ell^k : \ell \leq j, k \in K_j\}$ so that $g^\psi(i_\ell^k) = \psi(\ell)$ for all $k \in K_j$ (and otherwise agrees with g).

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By this process it is a simple matter to ensure that f_j^ψ forces a value on $\dot{h}(i_j^k)$ for all $k \in K_j$. (by the assumption that f forces that $\dot{h} \restriction \{i_j^k : k \in K_j\}$ is in V).

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We repeat the above fusion exactly except this time the definition of i_j^k is the maximum element of $\iota^{m_k} \setminus dom(f_{j-1})$ rather than the minimum.

And again, we finish the fusion, obtaining a larger function f and so that $\iota^{m_k} \setminus dom(f) \subset \{i_0^k, \ldots, i_j^k\}$ for some j (whose value diverges to infinity along some set K).

the new point x is not a 3 point

The construction has arranged that for each k and i^k_ℓ, and each function $s : S_k \mapsto 2$, each of $f \cup s \restriction (S_k \cap i^k_\ell + 1)$ and $f \cup s \restriction (S_k \setminus i^k_\ell)$ force a value on $\dot{h}(i)$. Since they can't be different values, it follows that the value of $s(i^k_\ell)$ is really what is determining $\dot{h}(i^k_\ell)$ (and we're done).

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With our condition f as above and $I = \mathbb{N} \setminus dom(f)$, we partition $I = I_0 \cup I_1$ by $i \in I_0$ iff $f \cup \{(i, 0)\} \Vdash \dot{h}(i) = 0$;

by symmetry may assume $\limsup |I_0 \cap S_k|$ is infinite. Then $f \cup I_1 \times \{1\}$ forces that $\dot{h}$ is constantly 0 on $A \setminus a^*_f$