# Moron Maps and subspaces of N*

## what you need to know if you want to work on N*

## and you should!

Alan Dow

Department of Mathematics
University of North Carolina Charlotte

winter school 2010

# Connecting Theme

Suppose that $f : \mathbb{N}^* \mapsto K$ is *precisely* 2-to-1 (distinct from $\leq$2-to-1). What can then be said of $K$ and $f$ (how $\mathbb{N}^*$-like is $K$?)

# Connecting Theme

Suppose that $f : \mathbb{N}^* \mapsto K$ is *precisely* 2-to-1 (distinct from $\leq$2-to-1). What can then be said of $K$ and $f$ (how $\mathbb{N}^*$-like is $K$?)

What are the results, what are the methods needed, and what are the connected questions along the way?

# Connecting Theme

Suppose that $f : \mathbb{N}^* \mapsto K$ is *precisely* 2-to-1 (distinct from $\leq$2-to-1). What can then be said of $K$ and $f$ (how $\mathbb{N}^*$-like is $K$?)

What are the results, what are the methods needed, and what are the connected questions along the way?

History to this question (of R. Levy): ?Glazer? and van Douwen's maximal space

# Connecting Theme

Suppose that $f : \mathbb{N}^* \mapsto K$ is *precisely* 2-to-1 (distinct from $\leq$2-to-1). What can then be said of $K$ and $f$ (how $\mathbb{N}^*$-like is $K$?)

What are the results, what are the methods needed, and what are the connected questions along the way?

History to this question (of R. Levy): ?Glazer? and van Douwen's maximal space

$E$ is a vD space if there is a 1-to-1 map $f : \mathbb{N} \mapsto E$ such that the extension $f = f^\beta : \beta\mathbb{N} \mapsto \beta E$ is $\leq$2-to-1; and such a space exists. And $\beta E$ can be embedded into $\beta\mathbb{N}$ so that $f$ is a retract.

# 2-to-1 maps

[vD] for each $y \in \beta E$, $|f^{-1}(y)| = 1$ iff $y$ is a **far point** of $E$ (not a limit of any countable (closed) discrete set).

# 2-to-1 maps

[vD] for each $y \in \beta E$, $|f^{-1}(y)| = 1$ iff $y$ is a **far point** of $E$ (not a limit of any countable (closed) discrete set).

**Question 1** Does every countable space have a far point? Does every vD space?

# 2-to-1 maps

[vD] for each $y \in \beta E$, $|f^{-1}(y)| = 1$ iff $y$ is a **far point** of $E$ (not a limit of any countable (closed) discrete set).

**Question 1** Does every countable space have a far point? Does every vD space?

Unfortunately, even if $E$ had no far points, $f \restriction \mathbb{N}^*$ is still 1-to-1 at the points of $f^{-1}(E)$. $\mathrm{MA}_{ctble}$ implies all countable spaces have far points.

# 2-to-1 maps

[vD] for each $y \in \beta E$, $|f^{-1}(y)| = 1$ iff $y$ is a **far point** of $E$ (not a limit of any countable (closed) discrete set).

**Question 1** Does every countable space have a far point? Does every vD space?

Unfortunately, even if $E$ had no far points, $f \restriction \mathbb{N}^*$ is still 1-to-1 at the points of $f^{-1}(E)$. $\mathrm{MA}_{ctble}$ implies all countable spaces have far points.

we could ask many questions about vD spaces, but the question is about 2-to-1 maps and images of $\mathbb{N}^*$ (not of $\beta\mathbb{N}$).

# 2-to-1 maps

[vD] for each $y \in \beta E$, $|f^{-1}(y)| = 1$ iff $y$ is a **far point** of $E$ (not a limit of any countable (closed) discrete set).

**Question 1** Does every countable space have a far point? Does every vD space?

Unfortunately, even if $E$ had no far points, $f \restriction \mathbb{N}^*$ is still 1-to-1 at the points of $f^{-1}(E)$. $\mathrm{MA}_{ctble}$ implies all countable spaces have far points.

we could ask many questions about vD spaces, but the question is about 2-to-1 maps and images of $\mathbb{N}^*$ (not of $\beta\mathbb{N}$). e.g. **Question 2** if $\mathbb{N}^*$ maps $\leq$ 2-to-1 onto $K \subset \mathbb{N}^*$, does the map lift to a ($\leq$ 2-to-1) map on(to) $\beta\mathbb{N}$?

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

1. is $K$ homeomorphic to $\mathbb{N}^*$?

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

1. is $K$ homeomorphic to $\mathbb{N}^*$?
2. is $f$ locally 1-to-1, i.e. $\mathbb{N}^* \oplus \mathbb{N}^* \mapsto \mathbb{N}^*$

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

1. is $K$ homeomorphic to $\mathbb{N}^*$?
2. is $f$ locally 1-to-1, i.e. $\mathbb{N}^* \oplus \mathbb{N}^* \mapsto \mathbb{N}^*$
3. is $f$ somewhere 1-to-1 (not irreducible)

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

1. is $K$ homeomorphic to $\mathbb{N}^*$?
2. is $f$ locally 1-to-1, i.e. $\mathbb{N}^* \oplus \mathbb{N}^* \mapsto \mathbb{N}^*$
3. is $f$ somewhere 1-to-1 (not irreducible)
4. is $K$ non-separable, non-ccc?

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

1. is $K$ homeomorphic to $\mathbb{N}^*$?
2. is $f$ locally 1-to-1, i.e. $\mathbb{N}^* \oplus \mathbb{N}^* \mapsto \mathbb{N}^*$
3. is $f$ somewhere 1-to-1 (not irreducible)
4. is $K$ non-separable, non-ccc?
5. are countable sets $C^*$-embedded?

# Levy's questions: Let $f : \mathbb{N}^* \mapsto K$ be 2-to-1

[Levy $\vdash$] countable discrete subsets of $K$ have closures homeomorphic to $\beta\mathbb{N}$. Hence $K$ has *weight* $\mathfrak{c}$.

1. is $K$ homeomorphic to $\mathbb{N}^*$?
2. is $f$ locally 1-to-1, i.e. $\mathbb{N}^* \oplus \mathbb{N}^* \mapsto \mathbb{N}^*$
3. is $f$ somewhere 1-to-1 (not irreducible)
4. is $K$ non-separable, non-ccc?
5. are countable sets $C^*$-embedded?

Item 3 is our starting point for investigation.

# Could $f$ be irreducible?

For each $a \subset \mathbb{N}$,

$f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$ is useful to consider

# Could $f$ be irreducible?

For each $a \subset \mathbb{N}$,

$f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$ is useful to consider

pull this back to $\mathbb{N}^*$:

Define $J_a = a^* \cap f^{-1}(f[(\mathbb{N} \setminus a)^*]$.

$J_a$ is homeomorphic to $J_{\mathbb{N} \setminus a}$ (via $f^{-1} \circ f$); and both to $f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$.

# Could $f$ be irreducible?

For each $a \subset \mathbb{N}$,

$f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$ is useful to consider

pull this back to $\mathbb{N}^*$:

Define $J_a = a^* \cap f^{-1}(f[(\mathbb{N} \setminus a)^*]$.

$J_a$ is homeomorphic to $J_{\mathbb{N}\setminus a}$ (via $f^{-1} \circ f$); and both to $f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$.

If $f$ is irreducible, each are nowhere dense.

then $\{J_a : a \in \mathcal{P}(\mathbb{N})\}$ is a covering of $\mathbb{N}^*$ by nwd sets, $n(\mathbb{N}^*) \leq \mathfrak{c}$

# Could $f$ be irreducible?

For each $a \subset \mathbb{N}$,

$f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$ is useful to consider

pull this back to $\mathbb{N}^*$:

Define $J_a = a^* \cap f^{-1}(f[(\mathbb{N} \setminus a)^*]$.

$J_a$ is homeomorphic to $J_{\mathbb{N} \setminus a}$ (via $f^{-1} \circ f$); and both to $f[a^*] \cap f[(\mathbb{N} \setminus a)^*] \subset K$.

If $f$ is irreducible, each are nowhere dense.

then $\{J_a : a \in \mathcal{P}(\mathbb{N})\}$ is a covering of $\mathbb{N}^*$ by nwd sets, $n(\mathbb{N}^*) \leq \mathfrak{c}$

this connects to studied questions about covering $\mathbb{N}^*$ by nwd sets

For example

For example

**Fact:** if vD spaces have “lots” of far points, then $J_A$ is a discrete weak P-set of $Z$

For example

**Fact:** if vD spaces have “lots” of far points, then $J_A$ is a discrete weak P-set of $Z$

**Question 3** Can $\mathbb{N}^*$ be covered by (discrete) [weak] P-sets?

For example

**Fact:** if vD spaces have “lots” of far points, then $J_A$ is a discrete weak P-set of $Z$

**Question 3** Can $\mathbb{N}^*$ be covered by (discrete) [weak] P-sets?
for weak P-sets, I only know "NO" if CH

For example

**Fact:** if vD spaces have “lots” of far points, then $J_A$ is a discrete weak P-set of $Z$

**Question 3** Can $\mathbb{N}^*$ be covered by (discrete) [weak] P-sets?
for weak P-sets, I only know "NO" if CH

**Question 4** Con(MA + no P-set cover) but PFA or MA$\vdash$?

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

e.g. put $a \in \mathcal{I}_f$ if $f$ is 2-to-1 and locally 1-to-1 on $a^*$; $a = b \cup c$, $f[b^*] = f[c^*] = K \setminus f[(\mathbb{N} \setminus a)^*]$

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

e.g. put $a \in \mathcal{I}_f$ if $f$ is 2-to-1 and locally 1-to-1 on $a^*$; $a = b \cup c$, $f[b^*] = f[c^*] = K \setminus f[(\mathbb{N} \setminus a)^*]$
we would say *trivially* 2-to-1 on $a^*$

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

e.g. put $a \in \mathcal{I}_f$ if $f$ is 2-to-1 and locally 1-to-1 on $a^*$; $a = b \cup c$, $f[b^*] = f[c^*] = K \setminus f[(\mathbb{N} \setminus a)^*]$
we would say *trivially* 2-to-1 on $a^*$

Set $K_1 = K \setminus U_0$ and $X_1 = f^{-1}[K_1]$, hence $f : X_1 \mapsto K_1$ is 2-to-1 (and repeat)

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

e.g. put $a \in \mathcal{I}_f$ if $f$ is 2-to-1 and locally 1-to-1 on $a^*$; $a = b \cup c$, $f[b^*] = f[c^*] = K \setminus f[(\mathbb{N} \setminus a)^*]$
we would say *trivially* 2-to-1 on $a^*$

Set $K_1 = K \setminus U_0$ and $X_1 = f^{-1}[K_1]$, hence $f : X_1 \mapsto K_1$ is 2-to-1 (and repeat)

think of $\mathbb{N}^*$ as $A_0 \oplus_{X_1} B_0$, each $A_0 \setminus X_1$ and $B_0 \setminus X_1$ mapping 1-to-1 onto $U_0$ (hence essentially to each other)

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

e.g. put $a \in \mathcal{I}_f$ if $f$ is 2-to-1 and locally 1-to-1 on $a^*$; $a = b \cup c$, $f[b^*] = f[c^*] = K \setminus f[(\mathbb{N} \setminus a)^*]$
we would say *trivially* 2-to-1 on $a^*$

Set $K_1 = K \setminus U_0$ and $X_1 = f^{-1}[K_1]$, hence $f : X_1 \mapsto K_1$ is 2-to-1 (and repeat)

think of $\mathbb{N}^*$ as $A_0 \oplus_{X_1} B_0$, each $A_0 \setminus X_1$ and $B_0 \setminus X_1$ mapping 1-to-1 onto $U_0$ (hence essentially to each other)

need a picture

# Back to 2-to-1: the CH story is very elegant

There is a dense open $U_0 \subset K$ such that $f$ is locally 1-to-1 on $f^{-1}[U_0]$ (stronger than somewhere 1-to-1)

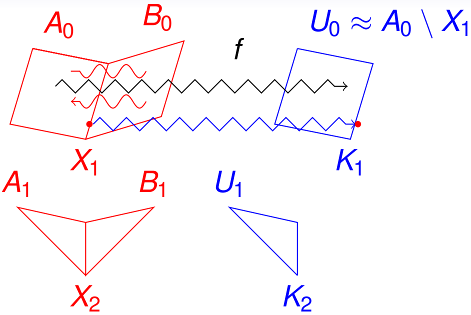
e.g. put $a \in \mathcal{I}_f$ if $f$ is 2-to-1 and locally 1-to-1 on $a^*$; $a = b \cup c$, $f[b^*] = f[c^*] = K \setminus f[(\mathbb{N} \setminus a)^*]$
we would say *trivially* 2-to-1 on $a^*$

Set $K_1 = K \setminus U_0$ and $X_1 = f^{-1}[K_1]$, hence $f : X_1 \mapsto K_1$ is 2-to-1 (and repeat)

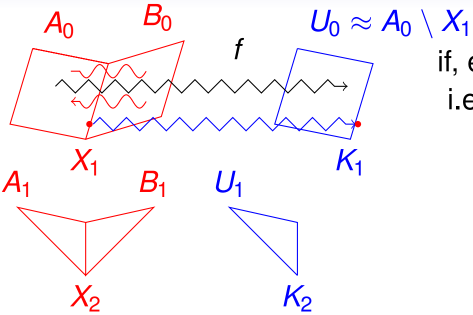
think of $\mathbb{N}^*$ as $A_0 \oplus_{X_1} B_0$, each $A_0 \setminus X_1$ and $B_0 \setminus X_1$ mapping 1-to-1 onto $U_0$ (hence essentially to each other)

need a picture

similarly there is $U_1 \subset K_1$ and $A_1 \oplus_{X_2} B_1$ with $X_2 = f^{-1}[K_2 = (K_1 \setminus U_1)]$

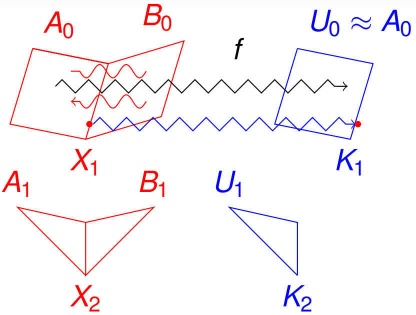



$X_1$
$= A_1 \cup B_1$




if, e.g. $K_2 = \emptyset$
i.e. $U_1 = K_1$

$X_1$
$= A_1 \cup B_1$

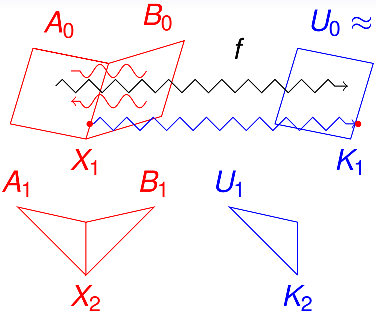



if, e.g. $K_2 = \emptyset$
i.e. $U_1 = K_1$

pick clopen set $W \subset \mathbb{N}^*$ such that $W \cap X_1 = A_1$

$X_1$
$= A_1 \cup B_1$

$U_0 \approx A_0 \setminus X_1$

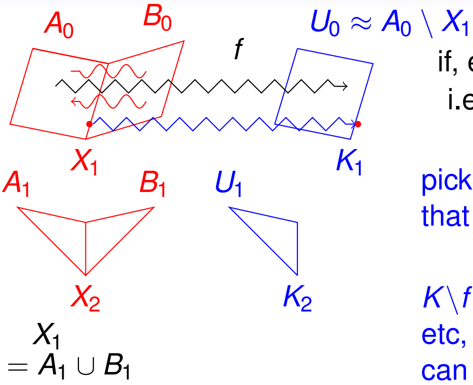



if, e.g. $K_2 = \emptyset$
i.e. $U_1 = K_1$

$X_1$
$= A_1 \cup B_1$

pick clopen set $W \subset \mathbb{N}^*$ such that $W \cap X_1 = A_1$

$K \setminus f[W]$ can be made clopen;
etc, etc, $K$ is Parovicenko
can be shown

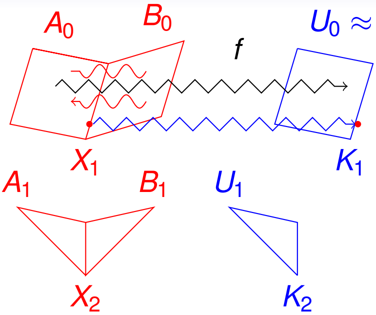



if, e.g. $K_2 = \emptyset$
i.e. $U_1 = K_1$

pick clopen set $W \subset \mathbb{N}^*$ such that $W \cap X_1 = A_1$

$K \setminus f[W]$ can be made clopen;
etc, etc, $K$ is Parovicenko
can be shown

$\vdash$ $K_n$ IS empty for some $n \in \omega$




$U_0 \approx A_0 \setminus X_1$

if, e.g. $K_2 = \emptyset$
i.e. $U_1 = K_1$

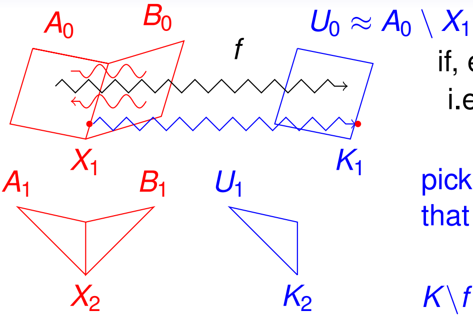
$X_1$
$= A_1 \cup B_1$

pick clopen set $W \subset \mathbb{N}^*$ such that $W \cap X_1 = A_1$

$K \setminus f[W]$ can be made clopen; etc, etc, $K$ is Parovicenko can be shown

can all this happen?

$\vdash$ $K_n$ IS empty for some $n \in \omega$




if, e.g. $K_2 = \emptyset$
i.e. $U_1 = K_1$

$X_1$
$= A_1 \cup B_1$

pick clopen set $W \subset \mathbb{N}^*$ such that $W \cap X_1 = A_1$

$K \setminus f[W]$ can be made clopen;
etc, etc, $K$ is Parovicenko
can be shown

can all this happen?

$\vdash$ $K_n$ IS empty for some $n \in \omega$

THUS CH implies $K \approx \mathbb{N}^*$

# tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$.

# tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$.

We could further measure $\tau(x) \geq k$ by increasing the number of wings

# tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$.

We could further measure $\tau(x) \geq k$ by increasing the number of wings

say that $x$ is a propeller point (?symmetric tie-point?) if $\mathbb{N}^* = A \oplus_x B$ and there is an autohomeomorphism $h$ such that $\{x\} = \text{fix}(h)$ and $h[A] = B$ (i.e. $h$ spins the propeller)

# tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$.

We could further measure $\tau(x) \geq k$ by increasing the number of wings

say that $x$ is a propeller point (?symmetric tie-point?) if $\mathbb{N}^* = A \oplus_x B$ and there is an autohomeomorphism $h$ such that $\{x\} = \text{fix}(h)$ and $h[A] = B$ (i.e. $h$ spins the propeller)

if $x_1, x_2$ are propeller points, then there is a 2-to-1 $f$ on $\mathbb{N}^*$ such that $K \approx A_1 \oplus_{x_2}^{x_1} B_2$, where

# tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$.

We could further measure $\tau(x) \geq k$ by increasing the number of wings

say that $x$ is a propeller point (?symmetric tie-point?) if $\mathbb{N}^* = A \oplus_x B$ and there is an autohomeomorphism $h$ such that $\{x\} = \text{fix}(h)$ and $h[A] = B$ (i.e. $h$ spins the propeller)

if $x_1, x_2$ are propeller points, then there is a 2-to-1 $f$ on $\mathbb{N}^*$ such that $K \approx A_1 \oplus_{x_2}^{x_1} B_2$, where
$\mathbb{N}^* = A_1 \oplus_{x_1} B_1$ and $\mathbb{N}^* = A_2 \oplus_{x_2} B_2$ witness the propellers

# tie-points and propeller points

Say that $x \in \mathbb{N}^*$ is a tie-point if there are closed sets $A, B$ covering $\mathbb{N}^*$ and $\{x\} = A' \cap B'$; denote this as $\mathbb{N}^* = A \oplus_x B$.

We could further measure $\tau(x) \geq k$ by increasing the number of wings

say that $x$ is a propeller point (?symmetric tie-point?) if $\mathbb{N}^* = A \oplus_x B$ and there is an autohomeomorphism $h$ such that $\{x\} = \mathrm{fix}(h)$ and $h[A] = B$ (i.e. $h$ spins the propeller)

if $x_1, x_2$ are propeller points, then there is a 2-to-1 $f$ on $\mathbb{N}^*$ such that $K \approx A_1 \oplus_{x_2}^{x_1} B_2$, where
$\mathbb{N}^* = A_1 \oplus_{x_1} B_1$ and $\mathbb{N}^* = A_2 \oplus_{x_2} B_2$ witness the propellers

I do not know if it's the same to ask for $x$ such that there is an involution $f$ on $\mathbb{N}^*$ with $\{x\} = \mathrm{fix}(f)$; but I think it is interesting to investigate possible "values" for $\mathrm{fix}(f)$

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$);

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq\mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq\mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

using $E = \{2n : n \in \omega\}$ and $O = \omega \setminus E$,

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq \mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

using $E = \{2n : n \in \omega\}$ and $O = \omega \setminus E$, we see that setting $A = (\omega \times (E \cup \{\omega\}))^*$ and $B = (\omega \times (O \cup \{\omega\}))^*$

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq\mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

using $E = \{2n : n \in \omega\}$ and $O = \omega \setminus E$, we see that setting $A = (\omega \times (E \cup \{\omega\}))^*$ and $B = (\omega \times (O \cup \{\omega\}))^*$ show that $\mathbb{N}^* \approx (\omega \times \{\omega\})^* \approx A \oplus_{\mathbb{N}^*} B$

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq \mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

using $E = \{2n : n \in \omega\}$ and $O = \omega \setminus E$, we see that setting $A = (\omega \times (E \cup \{\omega\}))^*$ and $B = (\omega \times (O \cup \{\omega\}))^*$
show that $\mathbb{N}^* \approx (\omega \times \{\omega\})^* \approx A \oplus_{\mathbb{N}^*} B$
with $\mathbb{N}^*$ as a propeller set, hence $K_2 \neq \emptyset$ (and iterate)

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq \mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

using $E = \{2n : n \in \omega\}$ and $O = \omega \setminus E$, we see that setting $A = (\omega \times (E \cup \{\omega\}))^*$ and $B = (\omega \times (O \cup \{\omega\}))^*$ show that $\mathbb{N}^* \approx (\omega \times \{\omega\})^* \approx A \oplus_{\mathbb{N}^*} B$ with $\mathbb{N}^*$ as a propeller set, hence $K_2 \neq \emptyset$ (and iterate)

also for any ultrafilter $\mathfrak{U} \in \mathbb{N}^*$, considering $(\omega \times \omega + 1)^*_{\mathfrak{U}}$ (the $\mathfrak{U}$-limits in the growth) exemplifies there is a propeller point

# propellers under CH and many copies of $\mathbb{N}^*$

Under CH, every point $x$ of $\mathbb{N}^*$ is a tie-point such that $\mathbb{N}^* = A \oplus_x B$ with, in addition, each of $A \approx B \approx \mathbb{N}^*$ (regular closed copies of $\mathbb{N}^*$); $x$ is a propeller point iff $x$ is a P-point.

for any compact 0-dim'l space $X$ of weight $\leq \mathfrak{c}$, (e.g. $X = \omega + 1$), $(\omega \times X)^*$ is homeomorphic to $\mathbb{N}^*$

using $E = \{2n : n \in \omega\}$ and $O = \omega \setminus E$, we see that setting $A = (\omega \times (E \cup \{\omega\}))^*$ and $B = (\omega \times (O \cup \{\omega\}))^*$ show that $\mathbb{N}^* \approx (\omega \times \{\omega\})^* \approx A \oplus_{\mathbb{N}^*} B$ with $\mathbb{N}^*$ as a propeller set, hence $K_2 \neq \emptyset$ (and iterate)

also for any ultrafilter $\mathfrak{U} \in \mathbb{N}^*$, considering $(\omega \times \omega + 1)^*_{\mathfrak{U}}$ (the $\mathfrak{U}$-limits in the growth) exemplifies there is a propeller point

my best guess for a $K \not\approx \mathbb{N}^*$ is to have propeller points $\mathbb{N}^* = A_i \oplus_{x_i} B_i$ so that $A_1 \not\approx \mathbb{N}^*$ and/or $A_1 \oplus^{x_1}_{x_2} B_2 \not\approx \mathbb{N}^*$

# some PFA tricks; tie-points; and regular closed sets

Can there be tie-points? and if there are, can $A \approx \mathbb{N}^*$?

# some PFA tricks; tie-points; and regular closed sets

Can there be tie-points? and if there are, can $A \approx \mathbb{N}^*$?

Major open problem: **Question 5** If $f$ embeds $\mathbb{N}^*$ into $\mathbb{N}^*$, is there a lifting from $\beta\mathbb{N}$ to $\beta\mathbb{N}$?

# some PFA tricks; tie-points; and regular closed sets

Can there be tie-points? and if there are, can $A \approx \mathbb{N}^*$?

Major open problem: **Question 5** If *f* embeds $\mathbb{N}^*$ into $\mathbb{N}^*$, is there a lifting from $\beta\mathbb{N}$ to $\beta\mathbb{N}$?

an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is *ccc over fin* if there is no uncountable almost disjoint family in $\mathcal{I}^+$;

# some PFA tricks; tie-points; and regular closed sets

Can there be tie-points? and if there are, can $A \approx \mathbb{N}^*$?

Major open problem: **Question 5** If $f$ embeds $\mathbb{N}^*$ into $\mathbb{N}^*$, is there a lifting from $\beta\mathbb{N}$ to $\beta\mathbb{N}$?

an ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is *ccc over fin* if there is no uncountable almost disjoint family in $\mathcal{I}^+$;

similarly a closed set $K \subset \mathbb{N}^*$ can be said to be ccc over fin if there is no uncountable family of disjoint clopen subsets of $\mathbb{N}^*$ each hitting $K$ (this is more general than requiring that $K$ is contained in a ccc space)

# the CH, Cohen + OCA tricks

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each “proper” extension)

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each "proper" extension)

[Farah?] ${}^{<\omega}2 \Vdash$ usually no harm done but might be useful

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each "proper" extension)

[Farah?] ${}^{<\omega}2 \Vdash$ usually no harm done but might be useful

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty,

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each "proper" extension)

[Farah?] ${}^{<\omega}2 \Vdash$ usually no harm done but might be useful

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$.

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each "proper" extension)

[Farah?] ${}^{<\omega}2 \Vdash$ usually no harm done but might be useful

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each "proper" extension)

[Farah?] ${}^{<\omega}2 \Vdash$ usually no harm done but might be useful

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

e.g. if $x_\alpha = (I_\alpha, J_\alpha) \in \mathcal{I} \times \mathcal{J}$ with $I_\alpha \cap J_\alpha = \emptyset$, and for $\alpha \neq \beta$, $(I_\alpha \cap J_\beta) \cup (J_\alpha \cap I_\beta) \neq \emptyset$,

# the CH, Cohen + OCA tricks

Let $\mathcal{I}, \mathcal{J}$ etc. be families from $\mathcal{P}(\mathbb{N})$

CH trick ${}^{<\omega_1}\omega_2 \Vdash$ if every $\aleph_1$-sized subcollection has a nice extension, then so must $\mathcal{I}, \mathcal{J}$ (in each "proper" extension)

[Farah?] ${}^{<\omega}2 \Vdash$ usually no harm done but might be useful

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

e.g. if $x_\alpha = (I_\alpha, J_\alpha) \in \mathcal{I} \times \mathcal{J}$ with $I_\alpha \cap J_\alpha = \emptyset$, and for $\alpha \neq \beta$, $(I_\alpha \cap J_\beta) \cup (J_\alpha \cap I_\beta) \neq \emptyset$, then $\overline{\bigcup_\alpha I^*_\alpha} \cap \overline{\bigcup_\alpha J^*_\alpha} \neq \emptyset$

# gaps and such

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

# gaps and such

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

e.g. if $x_\alpha = (I_\alpha, J_\alpha) \in \mathcal{I} \times \mathcal{J}$ with $I_\alpha \cap J_\alpha = \emptyset$, and for $\alpha \neq \beta$, $(I_\alpha \cap J_\beta) \cup (J_\alpha \cap I_\beta) \neq \emptyset$, then $\overline{\bigcup_\alpha I^*_\alpha} \cap \overline{\bigcup_\alpha J^*_\alpha} \neq \emptyset$

# gaps and such

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

e.g. if $x_\alpha = (I_\alpha, J_\alpha) \in \mathcal{I} \times \mathcal{J}$ with $I_\alpha \cap J_\alpha = \emptyset$, and for $\alpha \neq \beta$, $(I_\alpha \cap J_\beta) \cup (J_\alpha \cap I_\beta) \neq \emptyset$, then $\overline{\bigcup_\alpha I^*_\alpha} \cap \overline{\bigcup_\alpha J^*_\alpha} \neq \emptyset$

CH trick plus OCA trick implies no $(\omega_2, \kappa)$-gaps for $\kappa \notin \{1, \omega\}$

# gaps and such

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

e.g. if $x_\alpha = (I_\alpha, J_\alpha) \in \mathcal{I} \times \mathcal{J}$ with $I_\alpha \cap J_\alpha = \emptyset$, and for $\alpha \neq \beta$, $(I_\alpha \cap J_\beta) \cup (J_\alpha \cap I_\beta) \neq \emptyset$, then $\overline{\bigcup_\alpha I^*_\alpha} \cap \overline{\bigcup_\alpha J^*_\alpha} \neq \emptyset$

CH trick plus OCA trick implies no $(\omega_2, \kappa)$-gaps for $\kappa \notin \{1, \omega\}$

or if each $x_\alpha = h_\alpha : \mathbb{N} \mapsto \mathbb{N}$ is a partial function and $h_\alpha \cup h_\beta$ is not a function for $\alpha \neq \beta$, then there is no common mod finite extension

# gaps and such

OCA trick: If $\mathbb{X} \subset \mathcal{P}(\mathbb{N})$ and $R \subset [\mathcal{P}(\mathbb{N})]^2$ is open [often simply $x \cap y \neq \emptyset$], and if $\neg\exists \mathbb{X} = \bigcup_n \mathbb{X}_n$ such that $\bigcup_n [\mathbb{X}_n]^2 \cap R$ is empty, then there is a proper poset $\mathbb{Q}_R$ forcing an $R$-homogeneous $\{x_\alpha : \alpha \in \omega_1\} \subset \mathbb{X}$. so, by PFA, such a sequence actually exists

e.g. if $x_\alpha = (I_\alpha, J_\alpha) \in \mathcal{I} \times \mathcal{J}$ with $I_\alpha \cap J_\alpha = \emptyset$, and for $\alpha \neq \beta$, $(I_\alpha \cap J_\beta) \cup (J_\alpha \cap I_\beta) \neq \emptyset$, then $\overline{\bigcup_\alpha I^*_\alpha} \cap \overline{\bigcup_\alpha J^*_\alpha} \neq \emptyset$

CH trick plus OCA trick implies no $(\omega_2, \kappa)$-gaps for $\kappa \notin \{1, \omega\}$

or if each $x_\alpha = h_\alpha : \mathbb{N} \mapsto \mathbb{N}$ is a partial function and $h_\alpha \cup h_\beta$ is not a function for $\alpha \neq \beta$, then there is no common mod finite extension

so if $\mathcal{H}$ is a coherent family of functions and $\{\mathrm{dom}(h) : h \in \mathcal{H}\}$ is a $P_{\omega_2}$-ideal, then THERE IS a common mod finite extension

# forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by ${}^{<\omega_1}\omega_2$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.

# forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by ${}^{<\omega_1}\omega_2$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.

let $Q$ be a ccc poset of cardinality $\omega_1$ and $\{\dot{Y}_\alpha : \alpha \in \omega_1\}$ enumerate all (nice) $Q$-names of subsets of $\mathbb{N}$.

# forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by ${}^{<\omega_1}\omega_2$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.

let $Q$ be a ccc poset of cardinality $\omega_1$ and $\{\dot{Y}_\alpha : \alpha \in \omega_1\}$ enumerate all (nice) $Q$-names of subsets of $\mathbb{N}$.

inductively (or otherwise) choose $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\} \subset V \cap \mathcal{P}(\mathbb{N})$, so that, for $\beta < \alpha$, $\Vdash_Q \dot{Y}_\beta \cap (c_\alpha \cup d_\alpha) \neq^* c_\alpha$
(if possible: make them $\subset^*$ increasing)

# forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by ${}^{<\omega_1}\omega_2$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.

let $Q$ be a ccc poset of cardinality $\omega_1$ and $\{\dot{Y}_\alpha : \alpha \in \omega_1\}$ enumerate all (nice) $Q$-names of subsets of $\mathbb{N}$.

inductively (or otherwise) choose $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\} \subset V \cap \mathcal{P}(\mathbb{N})$, so that, for $\beta < \alpha$, $\Vdash_Q \dot{Y}_\beta \cap (c_\alpha \cup d_\alpha) \neq^* c_\alpha$ (if possible: make them $\subset^*$ increasing)

then in the extension by $Q$, $(\alpha, \beta) \in R$ providing $(c_\alpha \cap d_\beta) \cup (d_\alpha \cap c_\beta) \neq \emptyset$ satisfies that $[\mathbb{X}']^2 \cap R$ is not empty for all uncountable $\mathbb{X}' \subset \mathbb{X} = \omega_1$

# forcing a gap from Shelah-Steprans

Start with PFA, use the CH trick to pass to the forcing extension by ${}^{<\omega_1}\omega_2$. This leaves $\mathcal{P}(\mathbb{N})$ unchanged.

let $Q$ be a ccc poset of cardinality $\omega_1$ and $\{\dot{Y}_\alpha : \alpha \in \omega_1\}$ enumerate all (nice) $Q$-names of subsets of $\mathbb{N}$.

inductively (or otherwise) choose $\{(c_\alpha, d_\alpha) : \alpha \in \omega_1\} \subset V \cap \mathcal{P}(\mathbb{N})$, so that, for $\beta < \alpha$, $\Vdash_Q \dot{Y}_\beta \cap (c_\alpha \cup d_\alpha) \neq^* c_\alpha$ (if possible: make them $\subset^*$ increasing)

then in the extension by $Q$, $(\alpha, \beta) \in R$ providing $(c_\alpha \cap d_\beta) \cup (d_\alpha \cap c_\beta) \neq \emptyset$ satisfies that $[\mathbb{X}']^2 \cap R$ is not empty for all uncountable $\mathbb{X}' \subset \mathbb{X} = \omega_1$ thus this is a freezable gap: no $Y$ such that $Y \cap (c_\alpha \cup d_\alpha) =^* c_\alpha$ for all $\alpha$.

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Remark: CH implies every closed nowhere dense set is a boundary of a regular closed set

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed.

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

**Lemma:** $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

**Lemma:** $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

let $\{a_n : n \in \omega\} \subset \mathcal{I}$ be pairwise disjoint;

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

**Lemma:** $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

let $\{a_n : n \in \omega\} \subset \mathcal{I}$ be pairwise disjoint;

for each $g \in \mathbb{N}^\omega$, let $L_g = \bigcup_n a_n \cap g(n)$.

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

**Lemma:** $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

let $\{a_n : n \in \omega\} \subset \mathcal{I}$ be pairwise disjoint;

for each $g \in \mathbb{N}^\omega$, let $L_g = \bigcup_n a_n \cap g(n)$.

since $\partial A$ is ccc over fin there is an $f \in \mathbb{N}^\omega$ so that $\partial A \cap (L_g \setminus L_f)^*$ is empty for all $g \in \mathbb{N}^\omega$

# ccc over fin boundaries; per 2-points and embeddings

Theorem: [PFA] boundaries of regular closed subsets are not ccc over fin

Let $A \subset \mathbb{N}^*$ be regular closed. so $\mathcal{I} \cup \mathcal{J}$ is dense, where $a \in \mathcal{I}$ if $a^* \subset A$ and $b \in \mathcal{J}$ if $b^* \cap A = \emptyset$

**Lemma:** $\partial A$ is ccc over fin implies $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

let $\{a_n : n \in \omega\} \subset \mathcal{I}$ be pairwise disjoint;

for each $g \in \mathbb{N}^\omega$, let $L_g = \bigcup_n a_n \cap g(n)$.

since $\partial A$ is ccc over fin there is an $f \in \mathbb{N}^\omega$ so that $\partial A \cap (L_g \setminus L_f)^*$ is empty for all $g \in \mathbb{N}^\omega$

so pick, for each $g$, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$.

# $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

# $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

we just picked, for each $g$, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$.

# $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

we just picked, for each $g$, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$.

the $L_g$'s range over a $P_{\omega_2}$-ideal so

# $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

we just picked, for each $g$, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$.

the $L_g$'s range over a $P_{\omega_2}$-ideal so

let $h : \mathbb{N} \setminus L_f \mapsto 2$ mod finite extend $h_g$ for all $g \in \mathbb{N}^\omega$
ccc argument we'll see later

# $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

we just picked, for each $g$, $h_g : L_g \setminus L_f \mapsto 2$ so that $h_g^{-1}(0) \in \mathcal{I}$ and $h_g^{-1}(1) \in \mathcal{J}$.

the $L_g$'s range over a $P_{\omega_2}$-ideal so

let $h : \mathbb{N} \setminus L_f \mapsto 2$ mod finite extend $h_g$ for all $g \in \mathbb{N}^\omega$
ccc argument we'll see later

with $b = h^{-1}(1)$ and $J \subset \omega$ such that $b \cap a_n$ is infinite for each $n$, we have that $\partial A \cap \left(b \cap \bigcup_{n\in J} a_n\right)^*$ is not empty;
since ccc over fin implies such a $J$ must be finite, we finish that each of $\mathcal{I}$ and $\mathcal{J}$ are P-ideals

# now is time for CH * Cohen * OCA trick

we continue with proof that $\partial A$ is not ccc over fin

# now is time for CH * Cohen * OCA trick

we continue with proof that $\partial A$ is not ccc over fin

let $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$ be disjoint pairs from $\mathcal{I} \times \mathcal{J}$ chosen so as to be cofinal.

# now is time for CH * Cohen * OCA trick

we continue with proof that $\partial A$ is not ccc over fin

let $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$ be disjoint pairs from $\mathcal{I} \times \mathcal{J}$ chosen so as to be cofinal.

the technique (again, later) produces a proper poset collection of names 1-to-1 $\dot{\rho} : 2^{<\omega} \mapsto \mathbb{N}$ and
$\{\alpha(f, \xi) : \xi \in \omega_1, f \in V \cap 2^\omega\} \subset \omega_1$

# now is time for CH * Cohen * OCA trick

we continue with proof that $\partial A$ is not ccc over fin

let $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$ be disjoint pairs from $\mathcal{I} \times \mathcal{J}$ chosen so as to be cofinal.

the technique (again, later) produces a proper poset collection of names 1-to-1 $\dot{\rho} : 2^{<\omega} \mapsto \mathbb{N}$ and $\{\alpha(f, \xi) : \xi \in \omega_1, f \in V \cap 2^\omega\} \subset \omega_1$

so that $\exists n = n(f, \xi, \eta), k = k(f, \xi, \eta)$ satisfying

$$\dot{\rho}(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

# now is time for CH * Cohen * OCA trick

we continue with proof that $\partial A$ is not ccc over fin

let $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$ be disjoint pairs from $\mathcal{I} \times \mathcal{J}$ chosen so as to be cofinal.

the technique (again, later) produces a proper poset collection of names 1-to-1 $\dot\rho : 2^{<\omega} \mapsto \mathbb{N}$ and $\{\alpha(f,\xi) : \xi \in \omega_1, f \in V \cap 2^\omega\} \subset \omega_1$

so that $\exists n = n(f,\xi,\eta), k = k(f,\xi,\eta)$ satisfying

$$\dot\rho(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

set $C_f = \{\rho(f \restriction k) : k \in \omega\} \subset \mathbb{N}$, and $\Gamma_f = \{\alpha(f,\xi) : \xi \in \omega_1\}$

# continued

# continued

fix any "generic" filter meeting $\omega_1$-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f, \xi), n(f, \xi, \eta), k(f, \xi, \eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$.

# continued

fix any "generic" filter meeting $\omega_1$-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f, \xi), n(f, \xi, \eta), k(f, \xi, \eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$.

so that for $f \in \mathcal{F}$, $\xi \neq \eta \in \omega_1$,

$$\rho(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

# continued

fix any “generic” filter meeting $\omega_1$-many dense subsets of this iteration of proper posets, so as to have some uncountable $\mathcal{F} \subset 2^{<\omega}$ and $\{(a_\alpha, b_\alpha) : \alpha \in \omega_1\} \subset \mathcal{I} \times \mathcal{J}$, and values for $\alpha(f, \xi), n(f, \xi, \eta), k(f, \xi, \eta)$ for all $f \in \mathcal{F}$ and $\xi \in \omega_1$.

so that for $f \in \mathcal{F}$, $\xi \neq \eta \in \omega_1$,

$$\rho(f \restriction k) = n \in (a_{\alpha(f,\xi)} \cap b_{\alpha(f,\eta)}) \cup (a_{\alpha(f,\eta)} \cap b_{\alpha(f,\xi)})$$

we obtain that $C_f^* \cap \partial A$ is non-empty for all $f \in \mathcal{F}$ because

$$\partial A \supset \overline{\bigcup_{\alpha \in \Gamma_f} (a_\alpha \cap C_f)^*} \cap \overline{\bigcup_{\alpha \in \Gamma_f} (b_\alpha \cap C_f)^*} \neq \emptyset$$

# okay, we freeze a gap

# okay, we freeze a gap

we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that $a_\alpha$'s increase)

# okay, we freeze a gap

we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that $a_\alpha$'s increase)

a pair $(\rho, H) \in Q$ if there is an $n$ with $\rho : 2^{<n} \overset{1-1}{\mapsto} \mathbb{N}$, and $H \in [\omega_1]^{<\omega}$ is such that

for each $\alpha \neq \beta \in H$, and each $t \in 2^n$, there is a $k < n$ with $\rho(t \restriction k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$

# okay, we freeze a gap

we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that $a_\alpha$'s increase)

a pair $(\rho, H) \in Q$ if there is an $n$ with $\rho : 2^{<n} \overset{1-1}{\mapsto} \mathbb{N}$, and $H \in [\omega_1]^{<\omega}$ is such that

for each $\alpha \neq \beta \in H$, and each $t \in 2^n$, there is a $k < n$ with $\rho(t \restriction k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$

assume $\{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q$; and that $H_\xi \cap H_\eta = H$ for all $\xi \neq \eta \in \omega_1$; and pairwise "isomorphic"

set $A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha$ and $B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha$

# okay, we freeze a gap

we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that $a_\alpha$'s increase)

a pair $(\rho, H) \in Q$ if there is an $n$ with $\rho : 2^{<n} \overset{1-1}{\mapsto} \mathbb{N}$, and $H \in [\omega_1]^{<\omega}$ is such that

for each $\alpha \neq \beta \in H$, and each $t \in 2^n$, there is a $k < n$ with $\rho(t \restriction k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$

assume $\{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q$; and that $H_\xi \cap H_\eta = H$ for all $\xi \neq \eta \in \omega_1$; and pairwise "isomorphic"

set $A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha$ and $B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha$ Let $I_0 = J_0 = \omega_1$, $S_0 = \{i : \exists^{\omega_1} \xi \in I_0 \;\; i \in A_\xi\}$ and $T_0 = \{i : \exists^{\omega_1} \eta \in J_0 \;\; i \in B_\xi\}$

# okay, we freeze a gap

we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that $a_\alpha$'s increase)

a pair $(\rho, H) \in Q$ if there is an $n$ with $\rho : 2^{<n} \overset{1-1}{\mapsto} \mathbb{N}$, and $H \in [\omega_1]^{<\omega}$ is such that

for each $\alpha \neq \beta \in H$, and each $t \in 2^n$, there is a $k < n$ with $\rho(t \restriction k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$

assume $\{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q$; and that $H_\xi \cap H_\eta = H$ for all $\xi \neq \eta \in \omega_1$; and pairwise "isomorphic"

set $A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha$ and $B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha$ Let $I_0 = J_0 = \omega_1$, $S_0 = \{i : \exists^{\omega_1} \xi \in I_0 \ \ i \in A_\xi\}$ and $T_0 = \{i : \exists^{\omega_1} \eta \in J_0 \ \ i \in B_\xi\}$

there is $i_0 \in S_0 \cap T_0$; set $I_1 = \{\xi \in I_0 : i_0 \in A_\xi\}$; $J_1 = \{\eta \in J_0 : i_0 \in B_\eta\}$

# okay, we freeze a gap

we have the gap $\{a_\alpha, b_\alpha : \alpha \in \omega_1\}$; mod finite increasing and $a_\alpha \cap b_\alpha$ empty. (enough that $a_\alpha$'s increase)

a pair $(\rho, H) \in Q$ if there is an $n$ with $\rho : 2^{<n} \overset{1-1}{\mapsto} \mathbb{N}$, and $H \in [\omega_1]^{<\omega}$ is such that

for each $\alpha \neq \beta \in H$, and each $t \in 2^n$, there is a $k < n$ with $\rho(t \restriction k) \in (a_\alpha \cap b_\beta) \cup (a_\beta \cap b_\alpha)$

assume $\{(\rho, H_\xi) : \xi \in \omega_1\} \subset Q$; and that $H_\xi \cap H_\eta = H$ for all $\xi \neq \eta \in \omega_1$; and pairwise "isomorphic"

set $A_\xi = \bigcap_{\alpha \in H_\xi \setminus H} a_\alpha$ and $B_\xi = \bigcap_{\alpha \in H_\xi \setminus H} b_\alpha$ Let $I_0 = J_0 = \omega_1$, $S_0 = \{i : \exists^{\omega_1} \xi \in I_0 \;\; i \in A_\xi\}$ and $T_0 = \{i : \exists^{\omega_1} \eta \in J_0 \;\; i \in B_\xi\}$

there is $i_0 \in S_0 \cap T_0$; set $I_1 = \{\xi \in I_0 : i_0 \in A_\xi\}$; $J_1 = \{\eta \in J_0 : i_0 \in B_\eta\}$ repeat $2^n$ times getting $\{i_t\}_{t \in 2^n}$