Indestructibility of Vopěnka Cardinals

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Vopěnka's Principle

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Given a first order signature Σ with at least one binary relation:

- For any proper class A of Σ-structures, there are $\mathcal{M}, \mathcal{N} \in A$ such that there is a non-trivial elementary embedding from $\mathcal{M}$ to $\mathcal{N}$.

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Definition

A cardinal κ is a Vopěnka cardinal if κ is inaccessible, and for every set $A \subset V_\kappa$ of cardinality κ of Σ-structures, there are $\mathcal{M}, \mathcal{N} \in A$ such that there exists a non-trivial elementary embedding from $\mathcal{M}$ to $\mathcal{N}$.

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i.e.
κ is inaccessible and

$$V_\kappa \models \text{ Vopěnka's Principle}$$

where "classes" are taken to be arbitrary subsets of V_κ.

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Theorem (Solovay, Reinhardt and Kanamori)

An inaccessible cardinal κ *is a Vopěnka cardinal if and only if, for every* $A \subseteq V_\kappa$*, there is an* $\alpha < \kappa$ *such that for every* η *strictly between* α *and* κ*, there is a* λ *strictly between* η *and* κ *and an elementary embedding*

$$j : \langle V_\eta, \in, A \cap V_\eta \rangle \to \langle V_\lambda, \in, A \cap V_\lambda \rangle$$

with critical point α*, such that* $j(\alpha) > \eta$*.*

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We call α as in the theorem *extendible below* κ *for* A*.*

Question:

Are Vopěnka cardinals consistent with other statements known to be independent of ZFC, assuming only that Vopěnka cardinals are themselves consistent? Statements like

- GCH
- existence of morasses
- a definable well-order on the universe
- etcetera

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One can obtain models for these statements by forcing.

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This j' is well-defined and elementary because

$$p \Vdash \varphi(\sigma_1, \ldots, \sigma_n) \qquad \text{iff} \qquad j(p) \Vdash \varphi(j(\sigma_1), \ldots, j(\sigma_n)).$$

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- If this is the case, we choose G in such a way that H will contain p, and our embedding will lift, as desired.

Note in particular that while we can choose G in such a way that the embedding is lifted, it does *not* follow that the embedding will lift for arbitrary choices of G.

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This gives us a lot of flexibility for manipulating names.

We shall prove:

Theorem

Let κ be a Vopěnka cardinal. Suppose $\langle P_\alpha \,|\, \alpha \leq \kappa \rangle$ is the reverse Easton iteration of $\langle \dot{Q}_\alpha \,|\, \alpha < \kappa \rangle$ where

- *for each $\alpha < \kappa$, $|\dot{Q}_\alpha| < \kappa$, and*
- *for all $\gamma < \kappa$, there is an η_0 such that for all $\eta \geq \eta_0$,*

$$\mathbb{1}_{P_\eta} \Vdash \dot{Q}_\eta \text{ is } \gamma\text{-directed-closed.}$$

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In particular, *every* choice of generic for P_κ yields an extension universe in which κ is Vopěnka.

Proof

Let G be P_κ-generic over V and consider a P_κ-name $\dot{A}$ for a subset A of V_κ.

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In fact we can go much further, arranging that each name σ used for an element of A is *very* nice:

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Let $\dot{A}$ be a name for a set of Σ-structures with ordinal domains. There is a name $\dot{A}'$ equivalent to $\dot{A}$ such that for for every $\langle \sigma, p \rangle \in \dot{A}$,

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- *the names $\dot{E}^\sigma$ and $\dot{R}^\sigma$ involve no conditions larger than is necessary:*
  if δ is the least inaccessible cardinal greater than γ_σ such that $|P_\delta| \leq \delta$ and

  $$\zeta \geq \delta \rightarrow \Vdash_{P_\zeta} \dot{Q}_\zeta \text{ is } \gamma_\sigma^+\text{-directed-closed}$$

  then $\dot{R}^\sigma$ is a P_δ-name for a subset of γ_σ, and $\dot{E}^\sigma$ is a P_δ-name for a subset of γ_σ^2.

So assume that $\dot{A}$ is of this nice form from the Lemma, and let α be extendible below κ for $\dot{A}$ in V.

Let $\langle \sigma, q \rangle \in \dot{A}$ be such that $q \in G$ and σ^G is of rank greater than α.

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Let $j : \langle V_\eta, \in, \dot{A} \cap V_\eta \rangle \to \langle V_\lambda, \in, \dot{A} \cap V_\lambda \rangle$ in V witness that α is η-extendible below κ for $\dot{A}$. In particular, $j(\alpha) > \eta$.

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So by elementarity the support of $j(q)$ below $j(\alpha)$ is bounded below β.

Now since G_ξ is directed, $j``G_\xi$ is directed, so by $|P_\xi|^+$-directed-closure, there is a single condition $r$ in $P^\eta$ extending the tail (from $\alpha$ onward) of every element of $j``G_\xi$ — the master condition.

The conditions p and r have disjoint supports, so they are compatible, and their common extension "$p \cup r$" is a condition in P^ξ extending p that forces that $j \restriction V_\xi : V_\xi \to V_{j(\xi)}$ will lift to an elementary embedding $j' : V_\xi[G_\xi] \to V_{j(\xi)}[G_{j(\xi)}]$.

So we have shown that it is dense for $j \restriction V_\xi$ to lift; now we must use that to show that there is an elementary embedding between elements of A.

Since $\langle \sigma, q \rangle \in \dot{A}$, $\langle j(\sigma), j(q) \rangle \in \dot{A}$ by the assumption that j is elementary for structures incorporating $\dot{A}$. We assumed that $q \in G_\xi$, so the master condition forces that $j(\sigma)^G \in A$.

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By the definition of j', $j' \restriction \sigma^G$ is a map from σ^G to $j(\sigma)^G$, and it is elementary since j' is.

That is, $j' \restriction \sigma^G$ is elementary from σ^G to $j(\sigma)^G$, both of which are in A. □

Corollary

If the existence of a Vopěnka cardinal is consistent, then the existence of a Vopěnka cardinal is consistent with any of the following.

- *GCH*
- *A definable well-order on the universe.*
- *Morasses at every infinite successor cardinal.*