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Forcings which preserve large cardinals:

2. Failures of GCH.

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4. L-like universes and large cardinals.

Not covered: Forcings which use large cardinals, but destroy largeness (Singular Cardinal Hypothesis)

What are large cardinals?

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$\kappa > \aleph_0$

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κ is *measurable* iff:

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$\exists$ nonprincipal, κ-complete ultrafilter on κ

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Idea: κ is "large" iff κ is the critical point of an embedding
$j : V \longrightarrow M$ where M is "large"

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However: κ could be $H(\lambda)$-strong for all λ (i.e., the critical point of embeddings with arbitrary degrees of strength)

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Kunen: More than ω-superstrong is inconsistent
(cannot have $H(j^\omega(\kappa)^+) \subseteq M$

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Conclusion: We need large cardinals to show consistency.

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b. Cardinal characteristics at a measurable (new area)

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}$$

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d. Singular cardinal problems (Prikry-type forcings)

Forcings that preserve large cardinals: Silver lifting

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Remark: The lifting method is the most common, but *not* the only way to preserve large cardinals

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(d) $n+1$*-Hyperextender ultrapower uses* $f : H(j^n(\kappa)) \to V$*;* ω*-Hyperextender ultrapower uses* $f : H(j^\omega(\kappa)) \to V$*.*

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(d) $n+1$-Hyperextender ultrapower uses $f : H(j^n(\kappa)) \to V$; ω-Hyperextender ultrapower uses $f : H(j^\omega(\kappa)) \to V$.

Proof (a): Define $H = \{j(f)(a) \mid f : H(\kappa) \to V,\, a \in H(\lambda)\} \prec M$, $k : H \simeq M'$ the transitive collapse, $j' : V \to M'$ by $j' = k \circ j$. $\square$

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κ^+ *distributive forcing*

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Sometimes it is easy to lift $j : V \to M$ to $j^* : V[G] \to M[G^*]$.

Recall: $j : V \to M$ has critical point κ, G is P-generic over V, $P^* = j(P)$ and we want a G^* which is P^*-generic over M satisfying $j[G] \subseteq G^*$. We say that j *lifts for* P.

Small forcing

Suppose that P belongs to $H(\kappa)$ (P is small). Then j lifts for P.

Proof: $P^* = j(P) = P$. Take $G^* = G$. Then G^* is P^*-generic over $M \subseteq V$ and $j[G] = G \subseteq G^*$, trivially!

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P is κ^+ distributive iff the intersection of κ-many open dense sets is always nonempty.

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So P-lifting is nontrivial only when P has size at least κ and adds κ-sequences. A good example is κ-Cohen forcing.

An embedding which lifts for κ-Cohen?

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Easier lifting problem: $P =$ Cohen$(\kappa, 1)$, i.e. κ-Cohen forcing.
Bad news!

Theorem

Let P be κ-Cohen forcing. Then no $j : V \to M$ lifts for P.

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Solution: Force not just at κ, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$$

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Lift not just $P(\kappa) = \kappa$-Cohen forcing, but the entire iteration P ("Prepare below κ")

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Use *Easton support*, i.e., for p in $P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$, $\mathrm{Support}(p) = \{i \mid p \restriction i \nVdash p(\alpha_i) \text{ is trivial}\}$ has bounded intersection with each inaccessible.

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$$P(\leq \lambda) * P(> \lambda)$$

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where $P(\leq \lambda)$ has "size" λ and $P(> \lambda)$ is λ^+-closed (descending sequences of length λ have lower bounds). As in Easton's theorem, this gives cofinality preservation.

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Theorem

Assume GCH. Let $P = P(\leq \kappa) = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$ *be the iteration of* α*-Cohen for inaccessible* $\alpha \leq \kappa$ *described above.*

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Assume GCH. Let $P = P(\leq \kappa) = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$ *be the iteration of* α*-Cohen for inaccessible* $\alpha \leq \kappa$ *described above. Suppose that* $j : V \to M$ *is an extender ultrapower witnessing the* $H(\lambda)$*-strength of* κ *for some regular* λ *less than the least inaccessible above* κ*. Then* j *lifts for* P*.*

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Let $C(\leq \kappa) = C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)$ denote the P-generic and $V^* = V[C(\leq \kappa)]$

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$j^* : V[C(\leq \kappa)] \to M[C^*(\leq \kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))]$

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(a) $M^\kappa \cap V \subseteq M$.

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Therefore $P^*(\kappa, j(\kappa))$ is κ^+-closed in $V[C(\leq \kappa)]$.

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Therefore $P^*(\kappa, j(\kappa))$ is κ^+-closed in $V[C(\leq \kappa)]$. But then (b) and the λ^+ closure of $P^*(\kappa, j(\kappa))$ in $M[C(\leq \kappa)]$ implies that we can build a $P^*(\kappa, j(\kappa))$-generic in κ^+ steps.

Preparing κ-Cohen

Proof of (a): $M^\kappa \cap V \subseteq M$

Given $j(f_0)(a_0), j(f_1)(a_1), \cdots$ of length κ define $f : H(\kappa) \to V$ by $f(\langle x_0, x_1, \cdots \rangle = \langle f_0(x_0), f_1(x_1), \cdots \rangle$;

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So we have succeeded in lifting $j : V \longrightarrow M$ to
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Even worse, whereas before $j'[C(\kappa)]$ was equal to $C(\kappa)$, now $j'[C(\kappa)]$ is a complicated set of conditions!

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To extend j' further we want to find a generic $S^*(j(\kappa))$ for the Sacks($j(\kappa), j(\kappa^{++})$) of $M[S^*(<j(\kappa))]$ which contains $j'[S(\kappa)]$, where $S(\kappa)$ is the Sacks(κ, κ^{++})-generic, yielding:

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Conclusion: The fusion property for κ-Sacks is a good substitute for κ^+-distributivity, and therefore works better than κ-Cohen.

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Extends the tuning fork method form a κ-Sacks product to κ-Sacks iteration (of length κ^{++}).

Forcings that preserve large cardinals

(with Honzìk) (Special Case) Assume GCH and F is an Easton function such that $F \restriction \kappa$ is definable over $H(F(\kappa))$ uniformly for all regular κ. Then there is a cofinality-preserving forcing extension in which $2^\gamma = F(\gamma)$ for all regular γ and every κ which is $H(F(\kappa))$-strong in the ground model remains measurable.

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Cardinal characteristics at large cardinals

New area; we consider three examples:

$$\mathfrak{d}(\kappa),\ \mathrm{CofSym}(\kappa),\ \mathfrak{s}(\kappa)$$

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Cummings and Shelah proved an Easton-type theorem for the function $\kappa \mapsto \mathfrak{d}(\kappa)$. In particular:

Cardinal characteristics at large cardinals

Theorem

(Cummings-Shelah) Assume GCH and κ regular. Then in a cofinality-preserving extension, $\kappa^+ = \mathfrak{d}(\kappa) < 2^\kappa$.

Their proof goes as follows: First apply Cohen(κ, κ^{++}) to make $2^\kappa = \kappa^{++}$.

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We already saw the problems with lifting for Cohen(κ, κ^{++}); but κ-Hechler presents even more serious difficulties:

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But we have seen that the intersection of the $j(C)$, C club in κ is $\{\kappa\}$ and from this it follows that the $j(f)(\kappa)$ for $f : \kappa \to \kappa$ are cofinal in $j(\kappa)$.

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A forcing is κ^κ bounding iff every function $f : \kappa \to \kappa$ that it adds is dominated by such a function from the ground model. Any κ-cc forcing is κ^κ bounding, and fusion shows that Sacks(κ, κ^{++}) is also κ^κ bounding.

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Cardinal characteristics at large cardinals

With enough supercompactness, it can be shown that the κ-Cohen with κ-Hechler strategy does work, and indeed one can get κ measurable with any reasonable values for $\mathfrak{d}(\kappa)$, $\mathfrak{b}(\kappa)$ and 2^κ, where $\mathfrak{b}(\kappa)$ is the bounding number at κ, i.e., the smallest size of a subset of ${}^\kappa\kappa$ which is not bounded in ${}^\kappa\kappa$ under the order of eventual domination.

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Question: Is it consistent relative to a strong cardinal (i.e., a cardinal κ which is $H(\lambda)$-strong for all λ) to have a measurable κ with $\mathfrak{b}(\kappa) = \kappa^{++}$?

The Cardinal Characteristic $\mathrm{CofSym}(\kappa)$

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Macpherson and Neumann: $\mathrm{CofSym}(\kappa) > \kappa$

Cardinal characteristics at large cardinals

Sharp and Thomas: For any regular κ, can force $\mathrm{CofSym}(\kappa)$ to be greater than κ^+.

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(F-Zdomskyy) Suppose that κ is $H(\kappa^{++})$-strong. Then in a forcing extension, κ is measurable and $\mathrm{CofSym}(\kappa) = \kappa^{++}$.

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Question: Can one obtain a measurable κ with $\mathfrak{s}(\kappa) = \kappa^{++}$ from an α which is $H(\alpha^{++})$-strong?

Large Cardinals and L-like Universes

Question: Can we have the advantages of both $V = L$ and large cardinals?

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Outer model programme: A universe with large cardinals has an *outer model* which L-like and has large cardinals

1st approach uses fine structure theory and iterated ultrapowers
2nd approach uses forcing: much easier

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Examples of L-like properties:

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GCH

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Large Cardinals and L-like Universes: Forcing GCH

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Now we are done, as $P[\kappa, \infty)$ is κ^+-distributive and this implies that the image of $G[\kappa, \infty)$ generates a $P^*[j(\kappa), \infty)$-generic

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So we simply assume that $G(j(\kappa))$ was chosen below this greatest lower bound.

Finally in analogy to the superstrong case, the κ^{++}-distributivity of $P[\kappa^+, \infty)$ implies that the image of $G[\kappa^+, \infty)$ generates a $P^*[j(\kappa)^+, \infty)$-generic.

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Fact: If $j : V \longrightarrow M$ is 2-superstrong with $j^2(\kappa)$ least then j is continuous at $j(\kappa)$ and therefore $j^2(\kappa)$ has cofinality $j(\kappa)$.

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Fact: If $j : V \to M$ is 2-superstrong with $j^2(\kappa)$ least then j is continuous at $j(\kappa)$ and therefore $j^2(\kappa)$ has cofinality $j(\kappa)$.

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Large Cardinals and L-like Universes: Forcing GCH

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And the $j(\kappa)^+$-distributivity of $P[j(\kappa), \infty)$ implies that the image of $G[j(\kappa), \infty)$ generates a $P^*[j^2(\kappa), \infty)$-generic.

Finally, for the ω-superstrong case we choose $G(< j^\omega(\kappa))$ to contain a condition forcing $j[G(< j^n(\kappa))] \subseteq G(< j^{n+1}(\kappa))$ for each n, and show:

Claim. $G(< j^\omega(\kappa)) \cap P^*(< j^\omega(\kappa))$ is $P^*(< j^\omega(\kappa))$-generic over M.

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The proof of the Claim uses an argument regarding the "reduction" of dense sets.

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(Asperó-F) Preserving a proper class of ω-superstrongs it is possible to force GCH together with a wellorder of V whose restriction to $H(\kappa^+)$ is definable over $H(\kappa^+)$ for uncountable regular κ, uniformly.

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Question: With set-forcing, can one always add a definable wellorder of $H(\aleph_{\omega+1})$?

Large Cardinals and L-like Universes: Definable Wellorders

Note: One cannot expect to force a definable wellorder of $H(\omega_1)$; this is not possible if there is a proper class of Woodin cardinals, for example, as then Projective Determinacy holds in all set-generic extensions.

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Another note: It is definitely not always possible to force a definable wellorder of $H(\lambda^+)$ for singular λ:
This is contradicted by an elementary embedding from $L[H(\lambda^+)]$ to itself with critical point less than λ, using Kunen's proof that there is no nontrivial elementary embedding of V to itself.

Large Cardinals and L-like Universes: Forcing $\diamondsuit$

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Note that we can't set $G^*(j(\kappa)) = G(j(\kappa))$ as $j(\kappa)$ is in general singular in V, so $G(j(\kappa))$ is not even defined!

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Also M is κ-closed in V.
So we can build a $P^*(j(\kappa))$-generic in κ^+ steps, hitting the dense sets in $\mathcal{S}_i$ at step i.

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$\square$ asserts that one can assign CUB subsets C_α of ordertype $< \alpha$ to singular limit ordinals α which cohere: If $\bar{\alpha}$ is a limit point of C_α then $C_{\bar{\alpha}}$ is just an initial segment of C_α.

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Very similar to forcing $\diamondsuit$. At regular stage α force $\Box$ below α in the natural way. The main problem is to build $C(j(\kappa))$, as $j(\kappa)$ can be singular. Again the trick is to minimise $j(\kappa)$ so that it will have cofinality κ^+, enabling a construction of $C(j(\kappa))$ in κ^+ steps.

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Jensen's argument is essentially that if $\vec{C}$ witnesses $\Box_\kappa$ and $j : V \longrightarrow M$ witnesses hyperstrength, then there is a problem with the $\Box_{j(\kappa)}$-sequence $j(\vec{C})$ in M at the ordinal $\alpha = \sup \pi[\kappa^+]$.

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Theorem

(Cummings-F) (a) If κ is inaccessibly hyperstrong then $\square$ fails on the singular cardinals below κ.

Large Cardinals and L-like Universes: Forcing $\Box$

$\Box$ on the Singular Cardinals is also contradicted by large cardinals, but now the large cardinal strength is greater.
$j : V \longrightarrow M$ is *inaccessibly hyperstrong* iff $H(\lambda) \subseteq M$ for some inaccessible greater than κ; we say *almost inaccessibly hyperstrong* if λ is only required to be inaccessible in M.

Theorem

(Cummings-F) (a) If κ is inaccessibly hyperstrong then $\Box$ fails on the singular cardinals below κ.
(b) One can force $\Box$ on the singular cardinals preserving almost inaccessible hyperstrength.

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We also force *universal* morasses, which by an observation of Donder implies the consistency of "tree-like continuous scales" at very large cardinals.

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Peter Holy and I show that one can force this preserving ω-superstrongs; this is especially important when combined with some work of Neeman-Schimmerling:

Large Cardinals and L-like Universes: Condensation

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This gives a "quasi lower bound" on the consistency strength of PFA(c^+ linked).