Compactness of the order-sequential topology

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Outline

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- compactness: restrictions

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- towards compactness: KC spaces

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- questions

The order-sequential topology

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A sequence (a_n) in $\mathbb{B}$ *converges algebraically* to $a \in \mathbb{B}$ iff

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The *order-sequential topology* τ_s on $\mathbb{B}$ is the finest topology for which every algebraically convergent sequence in $\mathbb{B}$ is topologically convergent. (There *is* such a finest topology on $\mathbb{B}$: take the union of all such topologies as a subbase.)

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In fact, $(2^\kappa, \tau_s)$ is the *sequential modification* of $(2^\kappa, \tau_c)$; hence $(2^\kappa, \tau_s)$ is not compact unless $\kappa = \omega$.
The class of convergent sequences is the same.

Properties of the order-sequential topology:
[Balcar-Glowczynski-Jech]

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- τ_s is determined by the neighbourhood filter of zero.
- $(\mathbb{B}, \tau_s)$ has no isolated points unless $\mathbb{B}$ is finite.
- $(\mathbb{B}, \tau_s)$ is connected if $\mathbb{B}$ is complete and atomless.

Theorem (BGJ)

For a σ-complete algebra $\mathbb{B}$ the following are equivalent:

- *The topologically convergent sequences of $(\mathbb{B}, \tau_s)$ are precisely the algebraically convergent sequences of $\mathbb{B}$*
- *$\mathbb{B}$ is $(\omega, 2)$-distributive*
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Example

Enumerate $CO(2^\omega)$ as $\{a_n; n \in \omega\}$ and consider (a_n) as a sequence in the Cohen algebra $RO(2^\omega)$. It is easily seen that the sequence converges topologically to zero, while $\limsup a_n = 1$ and $\liminf a_n = 0$, so the sequence does not converge algebraically.

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Theorem (BGJ)

For a complete ccc algebra $\mathbb{B}$, the space $(\mathbb{B}, \tau_s)$ is Hausdorff if and only if $\mathbb{B}$ is a Maharam algebra: carries a strictly positive continuous submeasure.

Question: when is $(\mathbb{B}, \tau_s)$ compact?

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So looking for compact $(\mathbb{B}, \tau_s)$ besides $P(\omega)$, we have to drop T_2; that is, give up Maharamity.

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If not, let X be an antichain of size $\aleph_1$. By completeness, we can assume that X is a maximal antichain (add $-\bigvee X$ otherwise). This generates a copy of $P(\omega_1)$ as a complete subalgebra in $\mathbb{B}$; this copy is a sequentially closed, hence closed subspace of $(\mathbb{B}, \tau_s)$.

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If not, let X be an antichain of size $\aleph_1$. By completeness, we can assume that X is a maximal antichain (add $-\bigvee X$ otherwise). This generates a copy of $P(\omega_1)$ as a complete subalgebra in $\mathbb{B}$; this copy is a sequentially closed, hence closed subspace of $(\mathbb{B}, \tau_s)$. So if $(\mathbb{B}, \tau_s)$ is compact, then $(2^{\omega_1}, \tau_s)$ is a compact Hausdorff space, with a topology strictly finer than $(2^{\omega_1}, \tau_c)$ — a contradiction.

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We don't know of *any* ZFC examples.
Suslin is a candidate.

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Let $\mathbb{B}_\alpha$ be the subalgebra completely generated by T_α. $\mathbb{B}_\alpha$ is a copy of $P(\omega)$ and $\mathbb{B}_\alpha$ is a regular subalgebra of $\mathbb{B}_\beta$ for $\alpha < \beta < \omega_1$.

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The algebra $\mathbb{B}$ is a direct limit of the chain of algebras B_α.

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- Every $\mathbb{B}_\alpha$ is a closed nowhere dense subset of $\mathbb{B}_{\alpha+1}$.

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- Every $\mathbb{B}_\alpha$ is a closed nowhere dense subset of $\mathbb{B}_{\alpha+1}$.
- The space $(\mathbb{B}, \tau_s)$ is a direct limit of the spaces (B_α, τ_s).

Proof: consider a set $A \subseteq \mathbb{B}$ such that every $A \cap \mathbb{B}_\alpha$ is closed in $\mathbb{B}_\alpha$; show that A is sequentially closed in $\mathbb{B}$.

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Old spaces from *S. Franklin: Spaces where sequences suffice* work as the isolating examples.
Many of folklore results continue to hold for KC:

Lemma

A continuous bijection from a compact space to a KC space is a homeomorphism. A compact KC space is maximal compact and minimal KC.

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Minimal KC spaces are compact.

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Proof: Let τ be a strongly KC topology on $\mathbb{B}$ that is strictly coarser than τ_s. Then the identity mapping from $(\mathbb{B}, \tau_s)$ to $(\mathbb{B}, \tau)$ is a continuous bijection, hence a homeomorphism — a contradiction.

Question: is $\mathbb{B}(T)$ minimal KC?

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If not, see blackboard.

accumulation

Definition (Thümmel):
Let T be a Suslin tree. For a coloring $\chi : T \to 2$ and $\alpha < \beta < \omega_1$, say that β *returns to* α if there is an increasing sequence of ordinals $\alpha_n < \beta$ such that $\alpha_0 = \alpha$, $\sup \alpha_n = \beta$, and for every node $x \in T_\alpha$ there is a fixed color $k(x) \in 2$ with the property that for every $y \in T_\beta$ with $y > x$, the set $\{n \in \omega; \chi(y|\alpha_n) \neq k(x)\}$ is finite.

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The coloring *accummulates* if for some $\alpha < \omega_1$, there is unboundedly many $\beta > \alpha$ that return to α.
Fact: The level β returns to $\alpha < \beta$ if and only if the points $x_n = \bigvee \{p \in T_{\alpha_n}; \chi(p) = 1\} \in \mathbb{B}_{\alpha_n}$ converge to the point $x = \bigvee \{p \in T_\alpha; k(p) = 1\} \in \mathbb{B}_\alpha$ algebraically.

Theorem (Thümmel)

For a Suslin tree T, the following are equivalent.

1. *The space $(\mathbb{B}(T), \tau_s)$ is compact.*
2. *For every subtree $S \subseteq T$ of the form $S = \bigcup_{\alpha \in M} T_\alpha$, where $M \in [\omega_1]^{\omega_1}$, every coloring $\chi : S \to 2$ accummulates.*

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Conditions: "partial Suslin trees". Subtree $T \subseteq 2^{<\omega_1}$ of height $h(T) = \alpha + 1$, α limit; no uncountable antichains.

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For $\alpha \in \omega_1$, let D_α consist of those Jech trees $T \subseteq 2^{\omega_1}$ for which there is some β such that $\alpha + \beta < h(T)$ and for all $f \in T$, $\{\xi \in (\alpha, \beta); f(\xi) = 1\}$ is finite.

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Theorem

Every D_α is a dense set on the Jech forcing. The generic Suslin tree satisfies the accumulation condition for the inherited coloring.

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(2) In some literature, the notion of a *Suslin algebra* is more general: a complete ccc distributive algebra; not not necessarily a completion of a Suslin tree. (These consistently exist.) Can these be compact?
(3) Consistently (Jech) there is $2^{\aleph_1}$ isomorphism types of Suslin trees. Also consistently (Todorčević), $2^{\aleph_1}$ of them are rigid. Can this affect compactness?