Two sets

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Notation and Terminology

Let $(X, +)$ be any uncountable Polish abelian group and let $I \subseteq \mathscr{P}(X)$ s.t

- I is σ-ideal with a Borel base and
- I contains all singletons and
- I translation invariant.

The σ-ideal I is nice if has properties as above.
Let $\mathcal{B}_+(I) = Borel(X) \setminus I$ be set of all I-positive Borel sets.
$Perf(X)$ stands for set of all perfect subsets of X
In most part of presentation X is a real plane $\mathbb{R}^2$ and $+$ denotes adding vectors.

Definition (Cardinal coefficients)

Let X - Polish space and $I \subseteq \mathscr{P}(X)$ be σ ideal as above. Then for any $\mathscr{F} \subset I$ let

$$cov(\mathscr{F}, I) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathscr{F} \wedge \bigcup \mathscr{A} = X\}$$

$$cov_h(\mathscr{F}, I) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathscr{F} \wedge (\exists B \in \mathcal{B}_+(I)) \bigcup \mathscr{A} = B\}$$

Lines be the set of all lines in $\mathbb{R}^2$.
$\mathbb{L}$ σ-ideal of null sets and
$\mathbb{K}$ σ-ideal of all meager subsets of X.

Fact

$cov_h(Lines, \mathbb{L}) = 2^\omega$, $cov_h(Lines, \mathbb{K}) = 2^\omega$.

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Definition (Two-set)

A subset $X \subseteq \mathbb{R}^2$ of the real plane is a two-set iff meets every line in exactly two points.

Theorem (Mazurkiewicz 1914)

There exist a two-set.

Two-sets with a Hamel base

Definition

Let X be any uncountable Polish space. We say that a set $A \subseteq X$ is completely I-nonmeasurable iff

$$(\forall B \in \mathcal{B}_+(X))\, A \cap B \neq \emptyset \quad \wedge \quad B \cap A^c \neq \emptyset$$

Note that if $I = [X]^{\leq\omega}$ then A is Bernstein set. Moreover if $I = \mathbb{L}$ then A is completely nonmeasurable subset of X.

Theorem

Let $I \subseteq P(\mathbb{R}^2)$ be any nice σ-ideal with $cov_h(Lines, I) = 2^\omega$. Then there exists a two point set $A \subseteq \mathbb{R}^2$, that is completely I-nonmeasurable Hamel base.

Corollary

There exists a two point set $A \subseteq \mathbb{R}^2$, that is completely nonmeasurable Hamel base.

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Proof

Let $\{L_\xi : \xi < \mathfrak{c}\}$ all straight lines in the plane $\mathbb{R}^2$,
let $\{B_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all positive Borel sets in $\mathbb{R}^2$
$\{h_\xi : \xi < \mathfrak{c}\}$ be a Hamel base of $\mathbb{R}^2$ over $\mathbb{Q}$.
Define $\{A_\xi : \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ such that for every $\xi < \mathfrak{c}$,

1. $|A_\xi| < \omega$;
2. $\bigcup_{\zeta \leq \xi} A_\zeta$ does not have three collinear points;
3. $\bigcup_{\zeta \leq \xi} A_\zeta$ contains precisely two points of L_ξ;
4. $B_\xi \cap \bigcup_{\zeta \leq \xi} A_\zeta \neq \emptyset$;
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Then, the set $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ will have desired property.

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Marczewski ideal

Definition

If X is Polish space then $A \subseteq \mathbb{R}$ is s_0-Marczewski iff

$$(\forall P \in Perf(X))(\exists Q \in Perf(X)) \;\; Q \subseteq P \;\; \wedge \;\; Q \cap A = \emptyset$$

and $A \subseteq \mathbb{R}$ is s-Marczewski (s-measurable) iff

$$(\forall P \in Perf(X))(\exists Q \in Perf(X)) \;\; Q \subseteq P \wedge (Q \cap A = \emptyset \;\; \vee \;\; Q \subseteq A).$$

Theorem

There exists a two point set $A \subseteq \mathbb{R}^2$, that is s_0-Marczewski.

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Definition (Partial two-set)

We say that $A \subseteq \mathbb{R}^2$ is a partial two-set iff meets every line at most two times.

It is well known that the unit circle is a partial two-set which cannot be extended to two-set.

Theorem

There exists a two point set $A \subseteq \mathbb{R}^2$, that is s-nonmeasurable. Moreover A contains a subset of the unit circle of full outer measure.

Proof.

Let C be a unit circle, $Lines = \{l_\xi : \xi < \mathfrak{c}\}$ and $\mathcal{B}_+(C, \mathbb{L}) = \{P_\xi : \xi < \mathfrak{c}\}$. Define a sequences $\{A_\xi : \xi < \mathfrak{c}\}$ $\{y_\xi : \xi < \mathfrak{c}\}$ s.t. for every $\xi < \mathfrak{c}$

1. $|A_\xi| < \omega$;
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4. $P_\xi \cap \bigcup_{\zeta \leq \xi} A_\zeta \neq \emptyset$;
5. $y_\xi \in P_\xi$;
6. $A_\xi \cap \{y_\zeta : \zeta \leq \xi\} = \emptyset$.

Then $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ is required set. □

Iso-covering set

Definition (κ-set)

We say that $A \subseteq \mathbb{R}^2$ is an κ-set iff every line meets exactly in κ-points.

Definition (κ-iso cov)

We say that $A \subseteq \mathbb{R}^2$ is κ-iso cov set iff for every $X \in [\mathbb{R}^2]^\kappa$ there exist isometry g on the real plane such that $g[X] \subseteq A$.

Theorem

For $n \geq 2$ there exists n-set which is not 2-iso cov set

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Decomposition of two-sets

Theorem

Every two-set can be decomposed onto two bijections of the real line $\mathbb{R}$.

Theorem

There exists a null and meager two-set $A \subseteq \mathbb{R}^2$ s.t. every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ cannot be contained in A.

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There exists a null and meager two-set $A \subseteq \mathbb{R}^2$ s.t. every Baire measurable function $f : \mathbb{R} \to \mathbb{R}$ cannot be contained in A.

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Two-set vs. Luzin set

Fact

Any two-set cannot be

- *Bernstein set*
- *Luzin set and*
- *Sierpiński set.*

Proof.

1) Each line L is a perfect set such that $|A \cap L| = 2$, so A cannot be Bernstein.
2) Let M be a perfect meager subset of $\mathbb{R}$. Then $M \times \mathbb{R}$ is meager and $|(M \times \mathbb{R}) \cap A| = 2|M| = \mathfrak{c}$.
3) Let N be a perfect null subset of $\mathbb{R}$. Then $N \times \mathbb{R}$ is null and $|(N \times \mathbb{R}) \cap A| = 2|N| = \mathfrak{c}$. □

Theorem

Assume CH then

1. *there exists partial two point set A that is Luzin set,*
2. *there exists partial two point set B that is Sierpiński set.*

Partial two-sets with combinatorial properties

Definition (ad family)

The set $\mathcal{A} \subset [\omega]^{\omega}$ is almost disjoint family (ad) iff any two distinct members of $\mathcal{A}$ has finite intersection.

$\mathcal{A}$ is (mad) iff $\mathcal{A}$ is a maximal respect to the $\subseteq$.

Definition (Eventually different functions)

We say that $\mathcal{A} \subseteq \omega^{\omega}$ is eventually different family in Baire space ω^{ω} iff every two distinct members $x, y \in \mathcal{A}$ are equal only on the finite subset of the ω.

$\mathcal{A}$ is maximal eventually different family iff $\mathcal{A}$ is a maximal respect to the inclusion relation.

Theorem (CH)

Let $h : \mathbb{R} \to \omega^\omega$ be a standard Borel bijection. Then there exist the partial two-point set $A \subseteq \mathbb{R}^2$ on the real plane such that

$$\{h(\pi_i(x)) \in \omega^\omega : x \in A \wedge i \in \{0,1\}\ \} \text{ - max. eventually different.}$$

where π_i are projections onto i-th axis.

Remark

The same result is about mad family instead maximal evantually different functions family.

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Remark

The same result is about mad family instead maximal evantually different functions family.

Proof

Consider sequence $(M_\alpha : \alpha < \omega_1)$ the increrasing continuous chain of the countable subetst of $\mathbb{R}$ with $\mathbb{R} \subseteq \bigcup_{\alpha<\omega_1} M_\alpha$
Let us construct the transfinite sequence $(A_\alpha, F_\alpha) : \alpha < \omega_1$ s.t.

1. $(\forall \alpha < \omega_1)\ A_\alpha = \{x_\xi \in \mathbb{R}^2 :\ \xi < \alpha\} \in M_\alpha$ is a partially two-point set,
2. $(\forall \alpha < \omega_1)\ F_\alpha = \{h \circ \pi_i(x_\xi) :\ x_\xi \in A_\alpha \wedge i \in \{1, 2\}\}$ forms family eventually different functions,
3. $(\forall \alpha < \omega_1)(\forall u \in M_\alpha \cap (\omega^\omega \setminus F_\alpha))(\exists v \in F_{\alpha+1})\ |u \cap v| = \omega$.

.. Proof

Correctness: let us assume that A_α is build at $\alpha < \omega_1$ step.
Enumerate $(\omega^\omega \setminus F_\alpha) \cap M_\alpha = \{y_n : \ n \in \omega\}$.
In H_κ model we can to construct the sequence $x_n : n \in \omega$ as follows if $\{x_k : k < n\}$ is build then we can choose x_n such that

$$\text{for any } u \text{ if } u \in F_\alpha \cup \{x_k : k < n\} \text{ then}$$

$$|h(x_n) \cap u| < \omega \wedge (h^{-1}(x_n), h^{-1}(y_n)) \notin W_\alpha$$

where $W_\alpha = \{l \in \text{lines} : \ \ |l \cap Z_\alpha| = 2\}$ and

$$Z_\alpha = \{(x, y) : x, y \in h^{-1}[F_\alpha] \cup \{x_k : k < n\} \cup \{y_n : n \in \omega\} \wedge x \neq y\}.$$

Using properties (1), (2) and (3) it is easy to show that $A = \bigcup_{\alpha < \omega_1} A_\alpha$ fulfil the assertion of this Theorem.

Here we adopt the proof of the Kunen Theorem about existence of the indestructible mad family (see [Ku] for example).

Theorem

It is consistent with ZFC theory that $\neg CH$ and there exists partial two-set for which the image of the set of all coordinates forms the mad family size ω_1 by standard bijection $h : \mathbb{R} \to P(\omega)$.

Theorem

It is consistent that $\neg CH$ and

$$(\exists C \in [\mathbb{R}^2]^{\omega_2})(\exists A \in \mathbb{L})(\exists D_1 \in [C]^{\omega_1})$$

s.t

$$A + D_1 = \mathbb{R}^2 \wedge C \text{ is partial two-set.}$$

Moreover the set C is a Luzin set.

Proof

Let V - ground model with CH.
Now $\mathbb{P} = Fn(\omega_2, 2)$ be forcing adding indenpendetly $c_\alpha : \alpha < \omega_2$ Cohen points on the $\mathbb{R}^2$.
If $\alpha < \beta < \gamma < \omega_2$ then c_γ is Cohen over c_α and c_β.
Then $c_\gamma \notin l_{\alpha,\beta}$ where $c_\alpha, c_\beta \in l_{\alpha,\beta}$ forms line $l_{\alpha,\beta} \in \mathbb{K}$.
We see that $C = \{c_\alpha : \alpha < \omega_2\}$ is partial two set.

C is Luzin:

Let G be $\mathbb{P}$-generic ultrafilter over V.
Take $x \in \omega^\omega \cap V[G]$ be any Borel code for a meager subset of $\mathbb{R}^2$.
Find $I \in [\omega_2]^\omega$ and nice name $\tilde{x} \in V^{Fn(I,2)}$ for x.
Define
$G_I = \{p \in Fn(I,2) : p \in G\}$ $G_{\omega_2 \setminus I} = \{p \in Fn(\omega \setminus I, 2) : p \in G\}$.
Then

- $V[G] = V[G_I][G_{\omega_2 \setminus I}]$
- $x \in V[G_I]$ and
- for any $\alpha \in \omega_2 \setminus I$ $c_\alpha \in V[G] \setminus V[G_I]$ is Cohen over $V[G_I]$.

Then $C \cap \#x \subseteq \{c_\alpha : \alpha \in I\}$ is countable.

... Proof

Consider a Marczewski decomposition $A \cup B = \mathbb{R}^2$ where $A \in \mathbb{L}$, $B \in \mathbb{K}$ and $A \cap B = \emptyset$.
Choose $D \in V[G] \cap [\omega_2]^{\omega_1}$ and $x \in V[G]$
Then by $c.c.c.$ of $Fn(\omega_2, 2)$ we have

- $\exists D_1 \in V \cap [\omega_2]^{\omega_1}\ D \subseteq D_1$ and
- $(\exists I \in [\omega_2]^{\omega})\ V[G] = V[G_I][G_{\omega_2 \setminus I}]$ and $x \in V[G_I]$
- $(\forall \alpha \in D \setminus I)\ c_\alpha \in A - \{x\}$

Then finally in $V[G]$ we have $\mathbb{R}^2 \subseteq A - C_{D_1 \setminus I}$ where $C_{D_1 \setminus I} = \{c_\alpha \in C : \ \alpha \in D_1 \setminus I\}$.

Thank You

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