Ramsey Classification Theorems and applications to the Tukey theory of ultrafilters

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Ramsey Theorem. For each $k, n \geq 1$ and coloring $c : [\omega]^k \to n$, there is an infinite $M \subseteq \omega$ such that c restricted to $[M]^k$ monochromatic.

Erdős-Rado Canonization Theorem. For each $k \geq 1$ and each equivalence relation E on $[\omega]^k$, there is an infinite $M \subseteq \omega$ such that $\mathsf{E} \restriction [M]^k$ is *canonical*,

i.e. $\mathsf{E} \restriction [M]^k$ is given by E_I^k for some $I \subseteq k$.

For $a, b \in [\omega]^k$, a E_I^k b iff $\forall i \in I$, $a_i = b_i$.

Note. The Erdős-Rado Theorem is a canonization theorem for the fronts (barriers) of the form $[\omega]^k$ on the Ellentuck space.

Def. $\mathcal{F} \subseteq [\omega]^{<\omega}$ is a *front* on $[\omega]^\omega$ iff (i) for each $X \in [\omega]^\omega$, there is an $a \in \mathcal{F}$ such that $a \sqsubset X$; and (ii) for $a, b \in \mathcal{F}$, $a \not\sqsubset b$.

Def. For a front $\mathcal{F}$, a map $\varphi : \mathcal{F} \to \mathbb{N}$ is *irreducible* if φ is

(a) *inner*, i.e. $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$, and

(b) *Nash-Williams*, i.e. for each $a, b \in \mathcal{F}$, $\varphi(a) \not\sqsubset \varphi(b)$.

Pudlak-Rödl Thm. For every front (barrier) $\mathcal{F}$ on $\mathbb{N}$ and every equivalence relation E on $\mathcal{F}$, there is an infinite $M \subseteq \mathbb{N}$ such that E $\restriction (\mathcal{F}|M)$ is represented by an irreducible mapping defined on $\mathcal{F}|M$.

Def. $\mathcal{F}|M = \{a \in \mathcal{F} : a \subseteq M\}$.

Let $l^0_0 = 0$, $l^0_1 = 1$, $l^0_2 = 3$, $l^0_3 = 6$, $\ldots$

$\mathbb{T}_1(0) = \{\langle\rangle, \langle 0\rangle\}$. $\mathbb{T}_1(n) = \{\langle\rangle, \langle n\rangle, \langle n, i\rangle : l^0_n \le i < l^0_{n+1}\}$, $n > 0$.

$\mathbb{T}_1 = \bigcup_{n<\omega} \mathbb{T}_1(n)$. (Draw $\mathbb{T}_1$ and $\mathbb{S}_1$.)

The space $(\mathcal{R}_1, \le_1, r^1)$

$X \in \mathcal{R}_1$ iff $X \subseteq \mathbb{T}_1$ and $X \cong \mathbb{T}_1$.

For $X, Y \in \mathcal{R}_1$, $Y \le_1 X$ iff $Y \subseteq X$.

$r^1_k(X) =$ the "initial segment" of X of length k.

$\mathcal{AR}^1_k = \{r^1_k(X) : X \in \mathcal{R}_1\}$. $\mathcal{AR}^1 = \bigcup_{k<\omega} \mathcal{AR}^1_k$.

- $\mathcal{R}_1$ comes immediately after the Ellentuck space in complexity.

The spaces $(\mathcal{R}_\alpha, \leq_\alpha, r^\alpha)$, $\alpha < \omega_1$.

Recursive Construction. Two cases: α is a successor ordinal, α is a limit ordinal. (Draw $\mathbb{T}_2$, $\mathbb{S}_2$, $\mathbb{T}_\omega$, $\mathbb{S}_\omega$.)

$X \in \mathcal{R}_\alpha$ iff $X \subseteq \mathbb{T}_\alpha$ and $X \cong \mathbb{T}_\alpha$.

For $X, Y \in \mathcal{R}_\alpha$, $Y \leq_\alpha X$ iff $Y \subseteq X$.

$r^\alpha_k(X) =$ the "initial segment" of X of length k.

$\mathcal{AR}^\alpha_k = \{r^\alpha_k(X) : X \in \mathcal{R}_1\}$. $\quad \mathcal{AR}^\alpha = \bigcup_{k<\omega} \mathcal{AR}^\alpha_k$.

Topological Ramsey spaces $(\mathcal{R}, \leq, r)$

basic open sets $[a, A] = \{X \in \mathcal{R} : \exists n(r_n(X) = a) \text{ and } X \leq A\}$.

Def. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey* iff for each $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that either (i) $[a, B] \subseteq \mathcal{X}$ or else (ii) $[a, B] \cap \mathcal{X} = \emptyset$.

Def. [Todorcevic] A triple $(\mathcal{R}, \leq, r)$ is a *topological Ramsey space* of every property of Baire subset of $\mathcal{R}$ is Ramsey and if every meager subset of $\mathcal{R}$ is Ramsey null.

Abstract Ellentuck Theorem. [Todorcevic]
If $(\mathcal{R}, \leq, r)$ satisfies **A.1 - A.4** and $\mathcal{R}$ is closed (in $\mathcal{AR}^{\mathbb{N}}$), then every property of Baire subset is Ramsey.

Classic Example. The Ellentuck space is a topological Ramsey space.

Thm. [D/T 2,3] For each $\alpha < \omega_1$, $(\mathcal{R}_\alpha, \leq_\alpha, r^\alpha)$ is a topological Ramsey space.

Rem 1. To each topological Ramsey space there correspond notions of selective and Ramsey ultrafilter. (They are not necessarily the same.) $\mathcal{R}_\alpha$ induces Laflamme's ultrafilter $\mathcal{U}_\alpha$.

Rem 2. $(\mathcal{R}_0, \leq_0, r^0)$ *is* the Ellentuck space.

Ramsey Classification Theorems for Fronts on $\mathcal{R}_\alpha$, $\alpha < \omega_1$

Def. An equivalence relation E on $\mathcal{AR}_k^\alpha$ is *canonical* iff it is induced by a downward closed subset of $r_k^\alpha(\mathbb{S}_\alpha)$.

Ramsey Classification Theorem for $\mathcal{AR}_k^\alpha$. [D/T 2,3]

Given $A \in \mathcal{R}_\alpha$, $k \geq 1$ and an equivalence relation E on $\mathcal{AR}_k^\alpha$, there is a $C \leq_\alpha A$ such that E $\restriction (\mathcal{AR}_k^\alpha | C)$ is canonical.

Numbers of Canonical Equivalence Relations.

$\mathcal{AR}_k^1$: $(2^1+1)(2^2+1)\cdots(2^k+1)$.

$\mathcal{AR}_1^2$: 4. $\mathcal{AR}_2^2$: $4 \cdot 6$. $\mathcal{AR}_3^2$: $4 \cdot 6 \cdot ((2^3+1)(2^4+1)+1)$.

$\mathcal{AR}_k^2$: $\Pi_{i<k}\left(\Pi_{l_i^1 \leq j < l_{i+1}^1}(2^j+1)+1\right)$.

Def. $\mathcal{F} \subseteq \mathcal{AR}^\alpha$ is a *front* on $\mathcal{R}_\alpha$ iff for each $Y \in \mathcal{R}_\alpha$, there is an $a \in \mathcal{F}$ such that $a \sqsubset Y$; and for $a, b \in \mathcal{F}$, $a \not\sqsubset b$.

Def. For $\mathcal{F}$ a front, a function φ on $\mathcal{F}$ is

1. *inner* if $\varphi(a) \subseteq a$ for all $a \in \mathcal{F}$.
2. *Nash-Williams* if $\varphi(a) \not\sqsubset \varphi(b)$, for all $a, b \in \mathcal{F}$.

Def. Let E be an equivalence relation on a front $\mathcal{F}$.

1. φ *represents* E iff for all $a, b \in \mathcal{F}$, $a\text{E}b$ iff $\varphi(a) = \varphi(b)$.
2. E is *canonical* iff E is represented by an inner Nash-Williams function φ, maximal among all inner Nash-Williams functions representing E.

Ramsey Classification Theorem for fronts on $\mathcal{R}_\alpha$. [D/T 2,3] Given any front $\mathcal{F}$ on $\mathcal{R}_\alpha$, $A \in \mathcal{R}_\alpha$ and equivalence relation E on $\mathcal{F}$, there is a $C \leq_\alpha A$ such that $\text{E} \restriction (\mathcal{F}|C)$ is canonical.

Motivation: Investigate the Structure of Tukey types of ultrafilters near the bottom of the Rudin-Keisler hierarchy

Def. $\mathcal{U} \geq_{RK} \mathcal{V}$ iff there is a function $h : \omega \to \omega$ such that $\mathcal{V} = h(\mathcal{U}) := \langle h''\mathcal{U} \rangle$.

Def. $\mathcal{U} \geq_T \mathcal{V}$ iff there is a *cofinal* map $h : \mathcal{U} \to \mathcal{V}$ taking cofinal subsets of $\mathcal{U}$ to cofinal subsets of $\mathcal{V}$.

$\mathcal{U} \geq_T \mathcal{V} \Rightarrow$ cof($\mathcal{U}$) $\geq$ cof($\mathcal{U}$) and add($\mathcal{U}$) $\leq$ add($\mathcal{V}$).

$\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$.

Fact. $\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U}$ and $\mathcal{V}$ are cofinally equivalent.

Fact. $\mathcal{U} \geq_{RK} \mathcal{V}$ implies $\mathcal{U} \geq_T \mathcal{V}$.

Thm. [Laflamme 89] For each $1 \leq \alpha < \omega_1$, there is a forcing $\mathbb{P}_\alpha = ([\omega]^\omega, \leq_{\mathbb{P}_\alpha})$ which adds a generic ultrafilter $\mathcal{U}_\alpha$ such that

1. $\mathcal{U}_\alpha$ is a rapid p-point satisfying certain partition properties.
2. The nonprincipal RK predecessors of $\mathcal{U}_\alpha$ form a decreasing chain of order type $(\alpha + 1)^*$, the least of which is Ramsey.
3. $\mathcal{U}_\alpha$ has complete combinatorics over $\text{HOD}(\mathbb{R})^{V[G]}$.

Question 1. What are the Tukey types below $\mathcal{U}_\alpha$?

Rem. $\mathcal{U}_1$ is weakly Ramsey but not Ramsey.

Thm. [Todorcevic in [Raghavan/Todorcevic]] If $\mathcal{U}$ is Ramsey and $\mathcal{V} \leq_T \mathcal{U}$, then $\mathcal{V}$ is isomorphic to some iterated Fubini product of $\mathcal{U}$.

Question 2. Is there some similar characterization of the ultrafilters $\leq_T \mathcal{U}_\alpha$ in terms of Rudin-Keisler?

Def. $\mathcal{U}$ is *Ramsey* iff for each function $c : [\omega]^2 \to 2$, there is a $U \in \mathcal{U}$ such that c is monochromatic on $[U]^2$.

Answers: Yes and Yes.

Fact. Each $\mathcal{U}_\alpha$ is a p-point.

Def. $\mathcal{U}$ is a *p-point* if for each sequence $X_0 \supseteq X_1 \supseteq \dots$ in $\mathcal{U}$, there is a $Y \in \mathcal{U}$ such that for each $n < \omega$, $Y \subseteq^* X_n$ (i.e. $|Y \setminus X_n| < \omega$).

Theorem. [D/T 1] If $\mathcal{U}$ is a p-point and $\mathcal{U} \geq_T \mathcal{W}$, then there is an $h : \mathcal{U} \rightarrow \mathcal{W}$ which is continuous, monotone, and cofinal.

Key Ideas. [D/T 2,3]

1. If $\mathcal{U}_\alpha \geq_T \mathcal{V}$, then there is a front $\mathcal{F}$ on $\mathcal{R}_\alpha$ and a function $f : \mathcal{F} \rightarrow \omega$ such that $f(\mathcal{U}_\alpha | \mathcal{F}) = \mathcal{V}$. This f induces an equivalence relation on $\mathcal{F}$.

2. The Ramsey Classification Theorem for $\mathcal{R}_\alpha$ gives understanding of the function f. Ramsey Theory is essential to this proof, which could not be done just by forcing.

The Tukey and Rudin-Keisler types Tukey below $\mathcal{U}_\alpha$

Thm. [D/T 2,3] For all $1 \leq \alpha < \omega_1$, there is a countable collection of rapid p-points $\mathcal{Y}_S^\alpha$ such that, if $\mathcal{V} \leq_T \mathcal{U}_\alpha$, then $\mathcal{V}$ is isomorphic to a tree ultrafilter (a countable iteration of Fubini products) of ultrafilters from among the $\mathcal{Y}_S^\alpha$.

The S are the downward closed subsets of $\mathbb{S}_\alpha(n)$, $n < \omega$, and $\mathcal{Y}_S^\alpha = \pi_S(\mathcal{U}_\alpha | \mathcal{R}_\alpha(n))$.

Thm. [D/T 2,3] For all $1 \leq \alpha < \omega_1$, the Tukey types of all ultrafilters Tukey reducible to $\mathcal{U}_\alpha$ form a decreasing chain of order type $(\alpha + 1)^*$. The Tukey least of these is a Ramsey ultrafilter.

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