RESOLVABILITY, PART 2.

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COROLLARY

COROLLARY (Pytkeev)

Assume that $\mu \geq \omega$ and for every $x \in X$ there is some $\kappa \geq \mu$ s.t. x is a B_κ-point. Then X is μ-resolvable.

PROOF. If τ is the topology of X, by Zorn's lemma there is a maximal topology $\varrho \supset \tau$ s.t. if $\mathcal{B}$ witnesses that (for some $\kappa \geq \mu$) x is a B_κ-point w.r.t. τ then the same is true w.r.t. ϱ.

Then $\langle X, \varrho \rangle$ is Pytkeev: If $Y \subset X$ is not ϱ-open then, by maximality, there is a B_κ-point (w.r.t. ϱ) $x \in Y$ and a witness $\mathcal{B}$ for this s.t. $B \setminus Y \neq \emptyset$ for all $B \in \mathcal{B}$. So, there is $Z \in [X \setminus Y]^{\leq \kappa}$ with $x \in \overline{Z}^{\varrho}$.

Thus $\langle X, \varrho \rangle$ is maximally resolvable, while $\Delta(X, \varrho) \geq \mu$, by definition. Consequently, X is μ-resolvable.

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PROBLEM. (Malychin, 1995)

Is a (hereditarily) Lindelöf T_3 space X with $\Delta(X) > \omega$ resolvable?

NOTE. Malychin constructed Hausdorff, and Pavlov even Uryson examples of Lindelöf irresolvable spaces.

THEOREM. (Filatova, 2004)

YES.

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Is a Lindelöf T_3 space X with $\Delta(X) > \omega$ 3-resolvable? ω-resolvable? or even maximally resolvable?

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Pavlov's theorem

$$s(X) = \sup\{|D| : D \subset X \text{ is discrete}\}$$
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THEOREM. (Pavlov, 2002)

(i) Any T_2 space X with $\Delta(X) > s(X)^+$ is maximally resolvable.
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THEOREM. (J-S-Sz, 2007)

If $\Delta(X) > s(X)$ then X is maximally resolvable. So, an HL space X with $\Delta(X) > \omega$ is maximally resolvable.

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Any T_3 space X with $e(X) = \omega < \Delta(X)$ is (2-)resolvable.

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Proof

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If X with $|X| \geq \kappa = \text{cf}(\kappa) > \omega$ has no right sep'd subset of size κ then X has a κ-resolvable subspace.

PROOF. May assume $X = \langle \kappa, \tau \rangle$. Set

$$T = \{x \in \kappa : \exists C_x \in \mathcal{C}(\kappa) \forall S \subset C_x \text{ non-st'ry } (x \notin \overline{S})\},$$
$$C = \Delta\{C_x : x \in T\} = \{\alpha < \kappa : \forall x \in \alpha \cap T \; (\alpha \in C_x)\} \in \mathcal{C}(\kappa).$$

Then $|T \cap C| < \kappa$, o.w. $\exists$ non-st'ry $S \in [C \cap T]^\kappa$ that is right sep'd:

$\forall x \in S,\ S \setminus (x+1) \subset C \setminus (x+1) \subset C_x$ implies $x \notin \overline{S \setminus (x+1)}$.

Thus T is non-st'ry and for $Y = \kappa \setminus T$ the ideal $\mathcal{I} = NS(Y)$ is κ-complete s.t. $\forall y \in Y \, \forall I \in \mathcal{I}$ there is $J \in \mathcal{I}$ with $I \cap J = \emptyset$ and $y \in \overline{J}$.

By the "ideal lemma", Y is κ-resolvable.

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If X with $|X| \geq \kappa = \mathrm{cf}(\kappa) > \omega$ has no right sep'd subset of size κ then X has a κ-resolvable subspace.

PROOF. May assume $X = \langle \kappa, \tau \rangle$. Set

$$T = \{x \in \kappa : \exists\, C_x \in \mathcal{C}(\kappa)\, \forall\, S \subset C_x \text{ non-st'ry } (x \notin \overline{S})\},$$
$$C = \Delta\{C_x : x \in T\} = \{\alpha < \kappa : \forall x \in \alpha \cap T\ (\alpha \in C_x)\} \in \mathcal{C}(\kappa).$$

Then $|T \cap C| < \kappa$, o.w. $\exists$ non-st'ry $S \in [C \cap T]^\kappa$ that is right sep'd:

$\forall x \in S,\ S \setminus (x+1) \subset C \setminus (x+1) \subset C_x$ implies $x \notin \overline{S \setminus (x+1)}$.

Thus T is non-st'ry and for $Y = \kappa \setminus T$ the ideal $\mathcal{I} = NS(Y)$ is κ-complete s.t. $\forall y \in Y\, \forall I \in \mathcal{I}$ there is $J \in \mathcal{I}$ with $I \cap J = \emptyset$ and $y \in \overline{J}$.

By the "ideal lemma", Y is κ-resolvable.

J-S-Sz

THEOREM. (J-S-Sz, 2007)

If $\Delta(X) \geq \kappa = \mathrm{cf}(\kappa) > \omega$ and X has no discrete subset of size κ then X is κ-resolvable.

PROOF. It suffices to show that X has a κ-resolvable subspace.

(i) If $Is(Y) \geq \kappa$ for every open $Y \subset X$ then X is κ-resolvable.

(ii) $Is(Y) < \kappa$ for some open $Y \subset X$, then there is $H \in [Y]^\kappa$ left sep'd. H has no right sep'd subset of size κ, as o.w. it had a discrete subset of size κ. So, H has a κ-resolvable subspace.

NOTE. Thus if $\Delta(X) > s(X)$ is regular then X is maximally resolvable. But if $\Delta(X) > s(X)$ is singular then the J-S-Sz-thm implies $< \Delta(X)$-resolvability only. However, in this case $\Delta(X) > s(X)^+$, so by Pavlov's theorem X is maximally resolvable.

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$< \lambda$-resolvable

THEOREM. (J-S-Sz, 2006)

For any $\kappa \geq \lambda = \mathrm{cf}(\lambda) > \omega$ there is a dense $X \subset D(2)^{2^\kappa}$ with $\Delta(X) = \kappa$ that is $< \lambda$-resolvable but not λ-resolvable.

NOTE. This solved a problem of Ceder and Pearson from 1967. We used the general method of constructing $\mathcal{D}$-forced spaces.

THEOREM. (Illanes, Baskara Rao)

If $\mathrm{cf}(\lambda) = \omega$ then every $< \lambda$-resolvable space is λ-resolvable.

PROBLEM.

Is this true for each singular λ? How about $\lambda = \aleph_{\omega_1}$?

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SD spaces

DEFINITION.
(i) $D \subset X$ is strongly discrete if there are pairwise disjoint open sets $\{U_x : x \in D\}$ with $x \in U_x$ for all $x \in D$.

(ii) X is an SD space if it is T_1 and every $x \in X$ is an SD limit.

EXAMPLE. In a T_3 space, every countable discrete set is strongly discrete.

THEOREM. (Sharma and Sharma, 1987)

Every SD space is ω-resolvable.

COROLLARY.

If the SD-limits are dense in a T_1 space X then X is (2-)resolvable.

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THEOREM. (J 2011)

Any T_3 space X with $\Delta(X) > e(X) = \omega$ is (2-)resolvable.

PROOF.

If $\Delta(X) > \omega_1 = e(X)^+$ then X is ω-resolvable by Pavlov's thm.

So assume $|X| = \Delta(X) = \omega_1$ and show that X has a resolvable subspace.

If $Y \subset X$ is open with $s(Y) = \omega$ then Y is ω_1-resolvable,
so can assume $s(Y) = \omega_1$ for Y open or regular closed.

Can assume that no point in X is an SD-limit (by Sh-Sh), hence every countable discrete subset of X is closed.

Then $S = \bigcup\{D' : D \in [X]^{\omega_1} \text{ is discrete }\}$ is dense in X.

NOTE: $D' = D^\circ$ for any discrete $D \in [X]^{\omega_1}$.

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Any T_3 space X with $\Delta(X) > e(X) = \omega$ is (2-)resolvable.

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If $\Delta(X) > \omega_1 = e(X)^+$ then X is ω-resolvable by Pavlov's thm.

So assume $|X| = \Delta(X) = \omega_1$ and show that X has a resolvable subspace.

If $Y \subset X$ is open with $s(Y) = \omega$ then Y is ω_1-resolvable, so can assume $s(Y) = \omega_1$ for Y open or regular closed.

Can assume that no point in X is an SD-limit (by Sh-Sh), hence every countable discrete subset of X is closed.

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LEMMA.

$\{D_n : n < \omega\} \subset [X]^{\omega_1}$ be discrete, $a \subset \omega$,

$$A = \bigcup_{n\in a} D_n \cup \bigcup_{n\in\omega\setminus a} D'_n \text{ and } B = \bigcup_{n\in a} D'_n \cup \bigcup_{n\in\omega\setminus a} D_n .$$

If $x \in S \setminus \overline{A}$ then there exists $D \in [X]^{\omega_1}$ discrete s.t. $x \in D'$ and

$$A \cap D' = \emptyset = B \cap D.$$

PROOF. By T_3, x has an open nbhd U s.t. $\overline{U} \cap \overline{A} = \emptyset$.
Then $\overline{U} \cap D'_n = \emptyset$, hence $|\overline{U} \cap D_n| \leq \omega$ for all $n \in \omega$.
Pick $E \in [U]^{\omega_1}$ discrete with $x \in E'$ and set $D = E \setminus \bigcup\{D_n : n \in \omega\}$.

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Write $S = \{x_\alpha : \alpha < \omega_1\}$ and define discrete $D_\alpha \in [X]^{\omega_1}$ with $x_\alpha \in D'_\alpha$ and sets $a_\alpha \subset \alpha + 1$ for $\alpha < \omega_1$ s.t. $\beta < \alpha$ implies $a_\beta = a_\alpha \cap (\beta + 1)$:

– Choose discrete $D_0 \in [X]^{\omega_1}$ with $x_0 \in D'_0$ and set $a_0 = \{0\}$.

– For $\alpha > 0$ set $a^-_\alpha = \bigcup\{a_\beta : \beta < \alpha\}$, moreover

$$A_\alpha = \bigcup\{D_\beta : \beta \in a^-_\alpha\} \cup \bigcup\{D'_\beta : \beta \in \alpha \setminus a^-_\alpha\},$$

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By induction, $A_\alpha \cap B_\alpha = \emptyset$ for all $\alpha < \omega_1$, hence $A = \bigcup\{A_\alpha : \alpha < \omega_1\}$ and $B = \bigcup\{B_\alpha : \alpha < \omega_1\}$ are (two) disjoint dense sets in X.

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