Successors of Singular Cardinals III: On the Schizophrenia of Jonsson Cardinals

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Definition

We say $\kappa \to [\kappa]^{<\omega}_{\kappa}$ if for any coloring c of the finite subsets of κ, there is an $H \subseteq \kappa$ of cardinality κ such that the range of $c \restriction [H]^{<\omega}$ is a proper subset of κ.

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A cardinal κ satisfying the above is called a Jonsson cardinal.

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- If κ carries a Jonsson algebra, so does κ^+. (Hence each $\aleph_n$ carries one.)
- It is unknown if $\aleph_\omega$ can be a Jonsson cardinal. We'll deal with $\aleph_{\omega+1}$ shortly.

Reformulation

A cardinal κ is Jonsson if and only if for every sufficiently large regular cardinal χ and every $x \in H(\chi)$, there is an elementary submodel M of $H(\chi)$ such that

- $x \in M$
- $\kappa \in M$,
- $|M \cap \kappa| = \kappa$, and
- $\kappa \not\subseteq M$.

Theorem

If κ is a regular Jonsson cardinal, then every stationary subset of κ reflects.

(Due to Woodin and Tryba independently.)

Let κ be a regular Jonsson cardinal, and suppose $M \prec H(\chi)$ (for some sufficiently large χ) satisfies

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It suffices to prove that every stationary $S \subseteq \kappa$ in M reflects. (Why?)

Let $S \in M$ be stationary in κ.

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Lemma

$S \setminus M$ is stationary.

(Blackboard)

Thus, we can find $\delta \in S \cap M$ such that $\delta = \sup(M \cap \delta)$.

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Note

- β_δ is a limit ordinal, and
- $\mathrm{cf}(\beta_\delta) > \aleph_0$.

(Why?)

We claim $S \cap \beta_\delta$ is stationary in δ.

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We claim that $\delta \in C$, and this yields a contradiction. (Blackboard)

Corollary

If κ is regular, then κ^+ carries a Jonsson algebra.

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Still open, but much is known. We will handle $\aleph_{\omega+1}$ next.

Scales

Definition

Let μ be a singular cardinal. A scale for μ is a pair $(\vec{\mu}, \vec{f})$ such that

- $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ is an increasing sequence of regular cardinals with supremum μ
- $\vec{f} = \langle f_\alpha : \alpha < \mu^+ \rangle$ is a sequence of functions such that
  - $f_\alpha \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$,
  - if $\alpha < \beta < \mu^+$ then $f_\alpha <^* f_\beta$ (modulo bounded)
  - if $f \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$, then $f <^* f_\alpha$ for some α.

Fundamental Fact

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This is a ZFC result, but we don't have much control over the sequence $\vec{\mu}$.

Theorem

One more fact... The following two statements are equivalent:

1. λ *is a Jonsson cardinal.*
2. *For every sufficiently large regular* $\chi > \lambda$*, whenever we are given a cardinal* κ *satisfying* $\kappa^+ < \lambda$*, there is an* $M \prec H(\chi)$ *such that*
   - $\{\lambda, \kappa\} \in M$,
   - $|M \cap \lambda| = \lambda$,
   - $\lambda \not\subseteq M$, *and*
   - $\kappa + 1 \subseteq M$.

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Suppose μ is singular, and $(\vec{\mu}, \vec{f})$ is a scale for μ for which each μ_i carries a Jonsson algebra. Then μ^+ carries a Jonsson algebra.

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Suppose not. Let $M \prec H(\chi)$ satisfy

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- $(\vec{\mu}, \vec{f}) \in M$,
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We must prove $\mu^+ \subseteq M$.

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A hint: What would happen if this failed? Why would a scale be useful?

Corollary

$\aleph_{\omega+1}$ *carries a Jonsson algebra.*

In general, if μ is singular and μ^+ is Jonsson, then no increasing sequence $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ consisting of successors of regular cardinals can be part of a scale for μ.

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This can be shown to imply that no collection of μ^+ sets in $[\mu]^{<\mu}$ can cover $[\mu]^{\mathrm{cf}(\mu)}$, and this in turn is enough to conclude ADS_μ holds.

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Schizophrenia. And it gets worse.

Theorem

If μ *is singular and* μ^+ *is Jonsson, then there is a proper ideal I on* μ^+ *such that*

- *I extends the non-stationary ideal*
- *I is* $\mathrm{cf}(\mu)$*-complete*
- *I is* θ*-indecomposable for all regular* θ *such that* $\mathrm{cf}(\mu) < \theta < \mu$ *(so I is closed under* increasing *unions of length* θ*)*
- *I is weakly* σ*-saturated for some* $\sigma < \mu$ *(so we cannot find* σ *disjoint I-positive sets).*

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Schizophrenia.

So what do these ideals look like? They come from club-guessing, and we'll look at one example.

Assume μ is singular of countable cofinality, and let $S = S^{\mu^+}_{\aleph_0}$.

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Further assume that for every closed unbounded $E \subseteq \mu^+$, there are stationarily many $\delta \in S$ such that $C_\delta \subseteq^* E$ (modulo finite).

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(Do such things actually exist?)

A set $A \subseteq \mu^+$ is in I if there is a club $E \subseteq \mu^+$ such that

$$\{\delta \in S : E \cap A \cap C_\delta \text{ is infinite}\} \text{ is non-stationary.} \tag{2}$$

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- I is an ideal extending the non-stationary ideal.
- If θ is an uncountable regular cardinal, then I is θ-indecomposable.
- If μ^+ is a Jonsson cardinal, then I is not weakly μ-saturated. (Actually, we can improve this, but this makes the point.)