On Successors of Singular Cardinals II

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Last Time

- If S is stationary, then Refl(S) means that every stationary subset of S reflects.
- $\square_\mu$ implies Refl(S) fails for every stationary $S \subseteq \mu^+$.
- If μ is singular and $\square_\mu$ fails, then $0^\sharp$ exists.

Current Project

Theorem

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Lemma

Suppose $\kappa < \lambda$ are regular cardinals, $S \subseteq S^\lambda_\kappa$ has no stationary initial segment, and A_δ is cofinal in δ of order-type κ for each $\delta \in S$. Then for each $\beta < \mu^+$, there is a regressive function F_β with domain $S \cap \beta$ such that the family $\{A_\alpha \setminus F_\beta(\alpha) : \alpha \in S \cap \beta\}$ is pairwise disjoint.

Proof of Theorem

Assume

- U is a uniform κ^+-complete ultrafilter on λ,
- $\langle A_\alpha : \alpha \in S\rangle$ is as in the assumptions of the lemma, and
- $\langle F_\beta : \beta < \mu^+\rangle$ is as in the conclusion of the lemma.

Proof of Theorem

Given $\alpha \in S$ and $\epsilon < \mu^+$, define B^α_ϵ to be those $\beta > \alpha$ for which $F_\beta(\alpha)$ is contained in the "first ϵ elements of A_α".

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Then

$$\bigcup_{\epsilon<\kappa} A^\alpha_\kappa = (\alpha, \lambda). \tag{1}$$

Hence there is $\epsilon(\alpha)$ such that $B^\alpha_{\epsilon(\alpha)} \in U$.

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Given $\alpha < \gamma$ in S, we know

$$B^{\alpha}_{\epsilon(\alpha)} \cap B^{\gamma}_{\epsilon(\gamma)} \neq \emptyset, \tag{2}$$

so choose β in both of these sets.

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Thus $A_\alpha \setminus F(\alpha)$ and $A_\gamma \setminus F(\gamma)$ are disjoint, hence F disjointifies $\{A_\alpha : \alpha \in S\}$.

This is impossible as S is stationary, hence Theorem 1 holds.

Remember

This implies the following statements:

1. If $\kappa < \lambda$ are regular with κ compact, then Refl($S^\lambda_{<\kappa}$) holds.
2. If μ is a singular limit of compact cardinals, then Refl(μ^+) holds.

Definition

ADS$_\mu$ means there is a family $\mathcal{A} = \langle A_\alpha : \alpha < \mu^+ \rangle$ of unbounded subsets of μ (not μ^+) such that $\langle A_\alpha : \alpha < \beta \rangle$ can be disjointified for each $\beta < \mu^+$.

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Note

- "ADS" stands for "almost disjoint sets".
- no subfamily of $\mathcal{A}$ of cardinality μ^+ can be disjointified.
- ADS_μ holds if μ is regular. (blackboard)

ADS$_\mu$, μ singular

What goes wrong if μ is singular?

Note: If ADS$_\mu$ holds for μ singular, then we may assume that each A_α is of order-type cf(μ).

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This simplifies the statements and proofs of theorems. In the end we will simply state the full results.

Restrictions on ultrafilters

Theorem (Theorem 2)

Suppose μ is singular of countable cofinality and ADS_μ holds. If I is a countably complete proper ideal on μ^+ containing the bounded ideal, then we can find μ^+ disjoint I-positive sets.

Proof of Theorem 2

Let $\langle A_\alpha : \alpha < \mu\rangle$ be an ADS$_\mu$-family, with each A_α of order-type ω, and let $\eta_\alpha : \omega \to A_\alpha$ be the increasing enumeration of A_α.

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For $\beta < \mu^+$, let F_β disjointify $\langle A_\alpha : \alpha < \beta \rangle$.

For $\alpha < \mu^+$ and $n < \omega$, define B_n^α be the set of $\beta > \alpha$ for which $F_\beta(\alpha) < \eta_\alpha(n)$.

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For $\beta < \mu^+$, let F_β disjointify $\langle A_\alpha : \alpha < \beta \rangle$.

For $\alpha < \mu^+$ and $n < \omega$, define B^α_n be the set of $\beta > \alpha$ for which $F_\beta(\alpha) < \eta_\alpha(n)$.

"The disjointer for β removes the first m elements of A_α for some $m < n$."

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Let $x_\alpha = \eta_\alpha(n(\alpha) + 1)$.

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Conclusion

$x_\alpha \in A_\alpha \setminus F_\beta(\alpha)$ for an I-positive set of β.

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Conclusion

There are μ^+ disjoint I-positive subsets of μ^+. Hence uniform countably complete filters on μ^+ are far from being ultrafilters.

Corollary

Suppose μ is singular of countable cofinality and there is a uniform countably-complete ultrafilter on μ^+. Then ADS$_\mu$ *fails.*

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Suppose μ is singular and ADS$_\mu$ *holds. If I is a proper* cf(μ)*-indecomposable ideal on μ^+ extending the bounded ideal, then there are μ^+ pairwise disjoint I-positive subsets of μ^+.*

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Corollary

If κ is compact, then ADS$_\mu$ *fails for every singular $\mu > \kappa$.*

Connection to cardinal arithmetic

Theorem

Suppose μ is singular of countable cofinality and $\kappa^{\aleph_0} < \mu$ for all $\kappa < \mu$. If $\mu^{\aleph_0} > \mu^+$, then ADS$_\mu$ holds.

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Suffices to find an “ADS$_\mu$-family” in *some* set A of cardinality μ.

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We construct $\{x_\alpha : \alpha < \mu^+\} \subseteq [\mu]^{\aleph_0}$ so that $\mathcal{A} = \{A_\alpha : \alpha < \mu^+\}$ witnesses ADS$_\mu$.

Lemma

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If $A \in \mathcal{F}$, then $|[A]^{\aleph_0}| < \mu$. So $\mathcal{F}$ can cover at most μ^+ elements of $[\mu]^{\aleph_0}$. But $\mu^{\aleph_0} > \mu^+$.

For $\beta < \mu^+$, fix a sequence $\langle A_n^\beta : n < \omega\rangle$ such that

- $A_0^\beta = \emptyset$
- $\beta = \bigcup_{n<\omega} A_n^\beta$
- $|A_n^\beta| < \mu$ for all $n < \omega$
- $A_n^\beta \subseteq A_{n+1}^\beta$.

By induction on $\alpha < \mu^+$, choose $x_\alpha \in [\mu]^{\aleph_0}$ such that for no $\beta < \mu^+$ and $n < \omega$ is x_α a subset of $\bigcup\{x_\gamma : \gamma \in A^\beta_n \cap \alpha\}$.

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Why is this possible? See Lemma 1.

This give us a family $\langle x_\alpha : \alpha < \mu^+ \rangle$.

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Let $\eta_\alpha : \omega \to x_\alpha$ be a bijection.

We want the family of sets of the form $\{\eta_\alpha \restriction \ell : \ell < \omega\}$ to witness ADS$_\mu$.

Given $\beta < \mu^+$, we need a function $h_\beta : \beta \to \omega$ such that

$$\Delta(\alpha, \gamma) \leq \max\{h_\beta(\alpha), h_\beta(\gamma)\} \tag{3}$$

for all $\alpha, \gamma < \beta$, where

$$\Delta(\alpha, \gamma) = \text{ least } \ell \text{ such that } \eta_\alpha(\ell) \neq \eta_\gamma(\ell). \tag{4}$$

Lemma

For each $n < \omega$, $\{x_\alpha : \alpha \in A^\beta_n\}$ has a one-to-one choice function f^β_n.

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We define $f^\beta_n \restriction (A^\beta_n \cap \alpha)$ be induction on α.

Lemma

For each $n < \omega$, $\{x_\alpha : \alpha \in A_n^\beta\}$ *has a one-to-one choice function* f_n^β.

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$\alpha = 0$ and α limit are trivial.

If $\alpha = \gamma + 1$, then x_γ is not a subset of $\bigcup\{x_\epsilon : \epsilon \in A^\beta_n \cap \gamma\}$ so we can define $f^\beta_n(\gamma)$.

Define $k_\beta : \beta \to \omega$ as follows:

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For $\alpha \in A^\beta_{n+1} \setminus A^\beta_n$, $k_\beta(\alpha)$ is the unique $k < \omega$ such that $f^\beta_n(\alpha) = \eta_\alpha(k)$.

Lemma

For fixed $\nu \in^{<\omega} \mu$, $\{\alpha < \beta : \nu = \eta_\alpha \restriction k_\beta(\alpha) + 1\}$ *contains at most one element of each* $A^\beta_{n+1} \setminus A^\beta_n$.

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Suppose $\alpha \neq \gamma$ in $A^\beta_{n+1} \setminus A^\beta_n$ and

$$\nu = \eta_\alpha \restriction (k_\beta(\alpha) + 1) = \eta_\gamma \restriction (k_\beta(\gamma) + 1). \tag{5}$$

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Then

$$f^\beta_n(\alpha) = \eta_\alpha(k_\beta(\alpha)) = \nu(k_\beta(\alpha)) = \nu(k_\beta(\gamma)) = \eta_\gamma(k_\beta(\gamma)) = f^\beta_n(\gamma). \tag{6}$$

Contradiction.

For $\alpha < \beta$, define

$$E(\alpha) = \{\gamma < \beta : \max\{k_\beta(\alpha), k_\beta(\gamma)\} < \Delta(\alpha, \gamma)\}. \qquad (7)$$

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$E(\alpha)$ consists of those γ for which k_β has failed to disjointify A_α and A_γ.

Lemma

$E(\alpha)$ is at most countable.

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If not, find k^* such that $B = \{\gamma \in E(\alpha) : k_\beta(\gamma) = k^*\}$ is uncountable. Set $\nu = \eta_\alpha \restriction k^* + 1$. Then for $\gamma \in B$, we have

$$\eta_\gamma \restriction k_\beta(\gamma + 1) = \eta_\alpha \restriction k^* = \nu, \tag{8}$$

contradicting the previous lemma.

Note that $\gamma \in E(\alpha)$ if and only if $\alpha \in E(\gamma)$, so we can define a graph Γ on β by connecting α and γ if and only if $\gamma \in E(\gamma)$.

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It is straightforward now to "correct" k_β to a function which works everywhere.

Corollary

If κ is compact, then the Singular Cardinals Hypothesis holds above κ.

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If μ is the least failure of SCH above κ, then ADS$_\mu$ holds by the preceding theorem. But ADS$_\mu$ cannot hold above a compact cardinal by our earlier work.