Marek Bienias

Independent Bernstein sets and algebraic constructions

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Introduction

Background

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Definition (Aron, Pérez-García and Seoane-Sepulvéda)

Let $\mathcal{L}$ be an algebra. A set $A \subseteq \mathcal{L}$ is said to be β-algebrable if there exists an algebra $\mathcal{B}$ so that $\mathcal{B} \subseteq A \cup \{0\}$ and $card(Z) = \beta$, where β is cardinal number and Z is a minimal system of generators of $\mathcal{B}$. Here, by $Z = \{z_\alpha : \alpha \in \Lambda\}$ is a minimal system of generators of $\mathcal{B}$, we mean that $\mathcal{B} = \mathcal{A}(Z)$ is the algebra generated by Z, and for every $\alpha_0 \in \Lambda$, $z_{\alpha_0} \notin \mathcal{A}(Z \setminus \{z_{\alpha_0}\})$. We also say that A is algebrable if A is β-algebrable for β-infinite.

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We study the following classes of functions:

- Perfectly everywhere surjective ($\mathcal{PES}$), strongly everywhere surjective ($\mathcal{SES}$) and everywhere discontinuous Darboux ($\mathcal{EDD}$) functions;
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Independent family of sets

Let $\mathcal{B}$ be a family of subsets of a set X. We say that the family $\mathcal{A}$ is $\mathcal{B}$-independent iff

$$A_1^{\varepsilon_1} \cap \ldots \cap A_n^{\varepsilon_n} \in \mathcal{B}$$

for any distinct $A_i \in \mathcal{A}$, any $\varepsilon_i \in \{0,1\}$ for $i \in \{1,...,n\}$ and $n \in \mathbb{N}$ where $A^0 = X \setminus A$ and $A^1 = A$.

There is an independent family of 2^κ many subsets of κ.

Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be a decomposition of $\mathbb{R}$ into disjoint Bernstein sets.

Let $\{N_\xi : \xi < 2^\mathfrak{c}\}$ be an independent family in $\mathfrak{c}$ such that for every $\xi_1 < ... < \xi_n < 2^\mathfrak{c}$ and for any $\varepsilon_i \in \{0,1\}$ the set $N_{\xi_1}^{\varepsilon_1} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n}$ is nonempty and has cardinality $\mathfrak{c}$.

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Independent family of Bernstein sets of cardinality $2^{\mathfrak{c}}$

For $\xi < 2^{\mathfrak{c}}$ put

$$B^{\xi} = \bigcup_{\alpha \in N_{\xi}} B_{\alpha}.$$

Then every set B^{ξ} is Bernstein. Note that for every $\xi_1 < \ldots < \xi_n < 2^{\mathfrak{c}}$ and any $\varepsilon_i \in \{0,1\}$ the set

$$(B^{\xi_1})^{\varepsilon_1} \cap \ldots \cap (B^{\xi_n})^{\varepsilon_n} = \bigcup_{\alpha \in N_{\xi_1}^{\varepsilon_1} \cap \ldots \cap N_{\xi_n}^{\varepsilon_n}} B_{\alpha}$$

is a Bernstein. That means $\{B^{\xi} : \xi < 2^{\mathfrak{c}}\}$ is the independent family of Bernstein sets.

Let for $\alpha < \mathfrak{c}$, $g_\alpha : B_\alpha \to \mathbb{C}$ (or $\mathbb{R}$) be a non-zero function. Let us put

$$f_\xi(x) = \begin{cases} g_\alpha(x) \text{ ,when } x \in B_\alpha \text{ and } \alpha \in N_\xi \\ 0 \text{ otherwise.} \end{cases}$$

Then the family $\{f_\xi : \xi < 2^{\mathfrak{c}}\}$ is linearly independent.

Remark

Let P be any non-zero polynomial without constant term and consider the function $P(f_{\xi_1}, ..., f_{\xi_n})$. Let

$$P_s(x) = P(\varepsilon_1 \cdot x, ..., \varepsilon_n \cdot x), s = (\varepsilon_1, ..., \varepsilon_n)$$

Let us observe here that the function $P(f_{\xi_1}, ..., f_{\xi_n})|_{B_\alpha}$ for any $\alpha \in N_{\xi_1}^{\varepsilon_1} \cap ... \cap N_{\xi_n}^{\varepsilon_n}$ is of the form

$$P(\varepsilon_1 \cdot g_\alpha, ..., \varepsilon_n \cdot g_\alpha) = P_s(g_\alpha)$$

Remark

Then we have two possibilities.

(i) Either at least one of the functions $P_s(x)$ for $s \in \{0,1\}^n$ is a non-zero polynomial of one variable. If P_s is non-zero, where $s = (\varepsilon_1, ..., \varepsilon_n)$, then the function $P(f_{\xi_1}, ..., f_{\xi_n})$ is non-zero on the Bernstein set of the form

$$(B^{\xi_1})^{\varepsilon_1} \cap (B^{\xi_2})^{\varepsilon_2} \cap ... \cap (B^{\xi_n})^{\varepsilon_n}.$$

(ii) Or every function of a type $P_s(x)$ is a zero function, and then $P(f_{\xi_1}, ..., f_{\xi_n})$ is zero function.

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$\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. The function $f : \mathbb{K} \to \mathbb{K}$ is called:

- perfectly everywhere surjective ($\mathcal{PES}(\mathbb{K})$) iff for every perfect set $P \subseteq \mathbb{K}, f(P) = \mathbb{K}$;
- strongly everywhere surjective ($\mathcal{SES}(\mathbb{K})$) iff it takes every real or complex value $\mathfrak{c}$ times on any interval.

The real function is an everywhere discontinuous Darboux function ($\mathcal{EDD}(\mathbb{R})$) iff it is nowhere continuous and maps connected sets to connected sets.

Proposition

Let $B \subseteq \mathbb{K}$ be a Bernstein set. There exist a function $f \in \mathcal{PES}(\mathbb{K})$ that is 0 on the set B^0.

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Let $B \subseteq \mathbb{K}$ be a Bernstein set and $\{P_\alpha : \alpha < \mathfrak{c}\}$ an ennumeration of all perfect sets in $\mathbb{K}$ and $\mathbb{K} = \{y_\beta : \beta < \mathfrak{c}\}$.

Then for every $\alpha < \mathfrak{c}$ cardinality of $B_\alpha = P_\alpha \cap B$ is continuum.

Ennumerate a product $\{B_\alpha : \alpha < c\} \times \{y_\beta : \beta < c\}$ as $\{A_\gamma : \gamma < \mathfrak{c}\}$, where $A_\gamma = (B_\gamma, y_\gamma)$.

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The following theorems hold and the proof is using a family of independent Berstein sets.

Theorem

The set $\mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable.

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The set $\mathcal{SES}(\mathbb{C})\setminus\mathcal{PES}(\mathbb{C})$ is $2^{\mathfrak{c}}$-algebrable.

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The set $\mathcal{EDF}(\mathbb{R})$ is $2^{\mathfrak{c}}$-algebrable but it is not strongly 1-algebrable.

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Let $C \subsetneq \mathbb{R}$ be a fixed closed subset of $\mathbb{R}$. We consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of C.

Theorem

The set of all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of C is $2^{\mathfrak{c}}$-algebrable.

Let $C \subsetneq \mathbb{R}$ be a fixed closed subset of $\mathbb{R}$. We consider functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous only in the points of C.

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proof (Sketch)

Let $[1,2] = \{r_\alpha : \alpha < \mathfrak{c}\}$ and
$g : \mathbb{R} \to \mathbb{R}$ be such that $g(x) = d(x, C)$. Then g is zero only on the set C.
Put $g_\alpha(x) = r_\alpha \cdot g(x)$ and f_ξ as in the general method.
If each function $P_s(x)$ is zero then $P(f_{\xi_1}, ..., f_{\xi_n})$ is zero function.
If $P_{s_0}(x)$ is non-zero for some $s_0 \in \{0,1\}^n$. Then $P(f_{\xi_1}, ..., f_{\xi_n})$ is continuous in any point of C and suppose that is continuous in a point $x_0 \notin C$.

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proof continued

$P(f_{\xi_1}, ..., f_{\xi_n})$ is zero on the Bernstein set

$$\bigcup_{\alpha \in N^0_{\xi_1} \cap N^0_{\xi_2} \cap ... \cap N^0_{\xi_n}} B_\alpha .$$

For every $\beta \in N^{\varepsilon_1}_{\xi_1} \cap N^{\varepsilon_2}_{\xi_2} \cap ... \cap N^{\varepsilon_n}_{\xi_n}$ there exist a sequence $(x_n)_{n \in \mathbb{N}} \subseteq B_\beta$ such that $x_n \rightarrow x_0$. Hence by the continuity of polynomial of one variable we get that $P_{s_0}(g_\beta(x_0)) = 0$ for any such β.

Since for $\alpha \neq \beta$ we have that $g_\alpha(x_0) = r_\alpha \cdot g(x_0) \neq r_\beta \cdot g(x_0) = g_\beta(x_0)$ so $P_{s_0}(g_\beta(x_0))$ as a polynomial of one variable β, that has infinitely many zeros, is zero function - contradiction.

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Question 1

Is the set $\mathcal{PES}(\mathbb{C})$ strongly $2^{\mathfrak{c}}$-algebrable? (answered 3 days ago)

Question 2

Is there a function $f \in \mathcal{EDC}(\mathbb{R})$ that has infinitely many values on each interval?

Question 3

Is the set $\mathcal{EDC}(\mathbb{R})$ strongly 1-algebrable (strongly $\mathfrak{c}$-algebrable, strongly $2^{\mathfrak{c}}$-algebrable)?

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Thank you for your attention :)