Remarks on ideal convergence

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Preliminaries

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A sequence $x \in \mathbb{R}^{\mathbb{N}}$ is called I-convergent to $t \in \mathbb{R}$ if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} \colon |x_n - t| \geq \varepsilon\} \in I$ (if such a t exists, then it is unique, and we write $t = I - \lim x$). By $c(I)$ we denote the set of all I-convergent sequences, and by $c_0(I)$ - the set of all sequences I-convergent to 0.

Refolmulated problem

In FGT, Remark 2 the following seminorm was introduced:

$$\|x\|_{\infty}^{I} = \inf\{\lambda > 0\colon \ (\exists K \in I^{*}) \ (\forall n \in K) \, |x_n| \leq \lambda\}.$$

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The question is, if $\ell^\infty(I)/c_0(I)$ is complete for any admissible I.

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Theorem

For every admissible ideal $I \subset P(\mathbb{N})$, the spaces $\ell^\infty(I)/c_0(I)$ and $C(P_I)$ are isometrically isomorphic. Consequently, $\ell^\infty(I)/c_0(I)$ is a Banach space.

Sketch of the proof

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Let us define a function $f_x : \mathbb{N} \cup P_I \to \mathbb{R}$ by the formula

$$f_x(p) = \begin{cases} x_p & \textit{for} \quad p \in \mathbb{N} \\ p^* - \lim x & \textit{for} \quad p \in P_I \end{cases}$$

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The function f_x is continuous (a topology in $\mathbb{N} \cup P_I$ is inherited from $\beta\mathbb{N}$).
Now we can define the required function $\Phi : \ell^\infty(I)/c_0(I) \to C(P_I)$:

$$\Phi([x]_\sim) = f_x | P_I.$$

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Definition

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$$Var(x_{|K}) = \sum_{n=1}^{\infty} |x_{k_n} - x_{k_{n+1}}| < \infty.$$

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We say that a sequence $x \in \mathbb{R}^{\mathbb{N}}$ is restrictively I-convergent, if there exist $\ell \in \mathbb{R}$ and a set $K = \{k_1 < k_2 < \dots\} \in I^*$ such that $\lim_{n\to\infty} x_{k_n} = \ell$.

The set of all such sequences we denote by $c^*(I)$.

Definition

We say that a sequence $x \in \ell^\infty(I)$ is I-monotone if there is a set $K \in I^*$ such that $x_{|K}$ is monotone in the usual way.

We denote all such sequences by $M(I)$.

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It is known [FGT] that

$$M(I) \subset W(I) \subset c^*(I) \subset c(I) \subset \ell^\infty(I).$$

Theorem

If $p = c$ then there is an admissible ideal I such that $M(I) = \ell^\infty(I)$.

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$$p = \min\{|A| \colon A \subset [\mathbb{N}]^{\mathbb{N}} \text{ has SFIP}, \neg(\exists X \in [\mathbb{N}]^{\mathbb{N}})(\forall Y \in A) X \subset^* Y\}.$$