Borel Conjecture and dual Borel Conjecture (and other variants of the Borel Conjecture)

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Outline of the talk

- Small sets of real numbers
  - Real numbers, topology, measure, algebraic structure
  - Meager, measure zero, strong measure zero, Borel Conjecture (BC)
- Sets which can be translated away from an ideal $\mathcal{J}$
  - $\mathcal{J}^\star$, strongly meager, dual Borel Conjecture (dBC)
  - Main theorem: Con(BC + dBC)
- Another variant of the Borel Conjecture
  - Marczewski ideal s_0, ${s_0}^\star$, "Marczewski Borel Conjecture" (MBC)
  - "Sacks dense ideals", perfectly meager sets, Con(MBC)?

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The real numbers: topology, measure, algebraic structure

The real numbers ("the reals")

- $\mathbb{R}$, the classical real line (connected, but not compact)
- $[0,1]$, the compact unit interval (connected, compact)
- ω^ω, the Baire space (totally disconnected, not compact)
- 2^ω, the Cantor space (totally disconnected, compact)
- $\mathcal{P}(\omega)$, equivalent to Cantor space via characteristic functions

Structure on the reals:

- natural topology (basic clopen sets/intervals form a basis)
- standard (Lebesgue) measure (equals length for intervals)
- group structure
  - e.g., $(2^\omega,+)$ is a topological group, with $+$ bitwise

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Two classical ideals: $\mathcal{M}$ and $\mathcal{N}$

$\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is an *ideal* if it is closed under subsets and finite unions; if an ideal is closed under countable unions, it is called σ-*ideal*.

A set $X \subseteq \mathbb{R}$ is *nowhere dense* if its closure has empty interior ($\overline{X}^\circ = \emptyset$). The nowhere dense sets form an ideal (but not a σ-ideal).

Definition

A set $X \subseteq \mathbb{R}$ is meager ($X \in \mathcal{M}$) iff it is contained in the union of countably many (closed) nowhere dense sets.

Both

- the family $\mathcal{M}$ of meager sets and
- the family $\mathcal{N}$ of Lebesgue measure zero sets

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Measure zero and strong measure zero sets

For an interval $I \subseteq \mathbb{R}$, let $\lambda(I)$ denote its length.

Definition (well-known)

A set $X \subseteq \mathbb{R}$ is (Lebesgue) measure zero ($X \in \mathcal{N}$) iff
for each positive real number $\varepsilon > 0$
there is a sequence of intervals $(I_n)_{n<\omega}$ of total length $\sum_{n<\omega} \lambda(I_n) \leq \varepsilon$
such that $X \subseteq \bigcup_{n<\omega} I_n$.

Definition (Borel; 1919)

A set $X \subseteq \mathbb{R}$ is strong measure zero ($X \in \mathcal{SN}$) iff
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- $\mathcal{SN} \subseteq \mathcal{N}$: each strong measure zero set is measure zero
- $[\mathbb{R}]^{\leq\omega} \subseteq \mathcal{SN}$: each countable set is strong measure zero
- $\mathcal{SN}$ is a translation-invariant σ-ideal
- A (non-empty) perfect set cannot be strong measure zero, hence
  - $\mathcal{SN} \subsetneq \mathcal{N}$ (think of the classical Cantor set $\subseteq [0,1]$)
  - there are no uncountable Borel sets in $\mathcal{SN}$

Question: Are there uncountable strong measure zero sets?

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The Borel Conjecture (BC)

Definition

The Borel Conjecture (BC) is the statement that there are **no** uncountable strong measure zero sets, i.e., $\mathcal{SN} = [\mathbb{R}]^{\leq\omega}$.

Proposition

CH (i.e., $2^{\aleph_0} = \aleph_1$) implies $\neg$BC.

Proof (Sketch).

- A Luzin set is an uncountable set whose intersection with any meager set is countable.
- Assuming CH, we can inductively construct a Luzin set.
- Every Luzin set is strong measure zero.

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The consistency of the Borel Conjecture

In 1976, Laver invented the method of *countable support forcing iteration* to prove Con(BC), the consistency of the Borel Conjecture:

Theorem (Laver; 1976)

There is a model of ZFC where the Borel Conjecture holds. More precisely, the Borel Conjecture can be obtained by a countable support iteration of Laver forcing of length ω_2.

Key points.

- it is necessary to add many dominating reals ("fast decreasing ε_n's")
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Equivalent characterization of strong measure zero sets

For $X, Y \subseteq \mathbb{R}$, let $X + Y = \{x + y : x \in X, y \in Y\}$.

Key Theorem (Galvin,Mycielski,Solovay; 1973)

A set $X \subseteq \mathbb{R}$ is strong measure zero if and only if for every meager set $M \in \mathcal{M}$, $X + M \neq \mathbb{R}$.

Note that $X + M \neq \mathbb{R}$ if and only if X can be "translated away" from M, i.e., there exists a $t \in \mathbb{R}$ such that $(X + t) \cap M = \emptyset$.

Proof of the easy direction.

- Given $(\varepsilon_n)_{n<\omega}$, let $D := \bigcup_{n<\omega}(q_n - \frac{\varepsilon_n}{2}, q_n + \frac{\varepsilon_n}{2})$ (q_n the rationals).
- D is dense, so $M := \mathbb{R} \setminus D$ is (closed) nowhere dense, hence meager.
- So there is a t such that $(X + t) \cap M = \emptyset$, so $(X + t) \subseteq D$.

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$\mathcal{J}$-shiftable sets ($\mathcal{J}^\star$)

Key Definition

Let $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$ be arbitrary. Define

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

$\mathcal{J}^\star$ is the collection of "$\mathcal{J}$-shiftable sets",
i.e., $Y \in \mathcal{J}^\star$ iff Y can be translated away from every set in $\mathcal{J}$.

Fact ("Galois connection")

Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(2^\omega)$ be arbitrary.

- $\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A}^\star \supseteq \mathcal{B}^\star$
- $\mathcal{A} \subseteq \mathcal{A}^{\star\star}$
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$$\mathcal{SN} = \mathcal{M}^\star$$

By replacing $\mathcal{M}$ by $\mathcal{N}$ we get a notion *dual to strong measure zero*:

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- $\mathcal{SM} \subseteq \mathcal{M}$: each strongly meager set is meager
  - the reals can be partitioned into a measure zero and a meager part
  - $Y \in \mathcal{SM}$ can be translated into the meager part of this partition
  - so the name is justified ;-)
- $[2^\omega]^{\leq\omega} \subseteq \mathcal{SM}$: each countable set is strongly meager
  - this is because $\mathcal{N}$ is a translation-invariant σ-ideal
- $\mathcal{SM}$ is translation-invariant, but (in general) it is NOT even an ideal

Question: Are there uncountable strongly meager sets?

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The dual Borel Conjecture (dBC)

Definition

The dual Borel Conjecture (dBC) is the statement that there are **no** uncountable strongly meager sets, i.e., $\mathcal{SM} = [2^\omega]^{\leq\omega}$.

Also dBC fails under CH. On the other hand, Carlson showed Con(dBC):

Theorem (Carlson; 1993)

The dual Borel Conjecture can be obtained by a finite support iteration of Cohen forcing of length ω_2.

Key points.

- Cohen reals are the canonical method to kill strongly meager sets.
- A strengthening of the c.c.c. ("precaliber $\aleph_1$") is used to avoid the resurrection of unwanted sets.

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The main theorem: Con(BC + dBC)

What about BC and dBC in the same model?

One of the obstacles in proving it:

- have to kill strongly meager sets to get the dual Borel Conjecture
- the standard way is adding Cohen reals
- but Cohen reals inevitably destroy the Borel Conjecture
- we have to kill strongly meager sets without adding Cohen reals
  - this is possible, but very difficult

Theorem (Goldstern, Kellner, Shelah, W.; 2011+ε)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).

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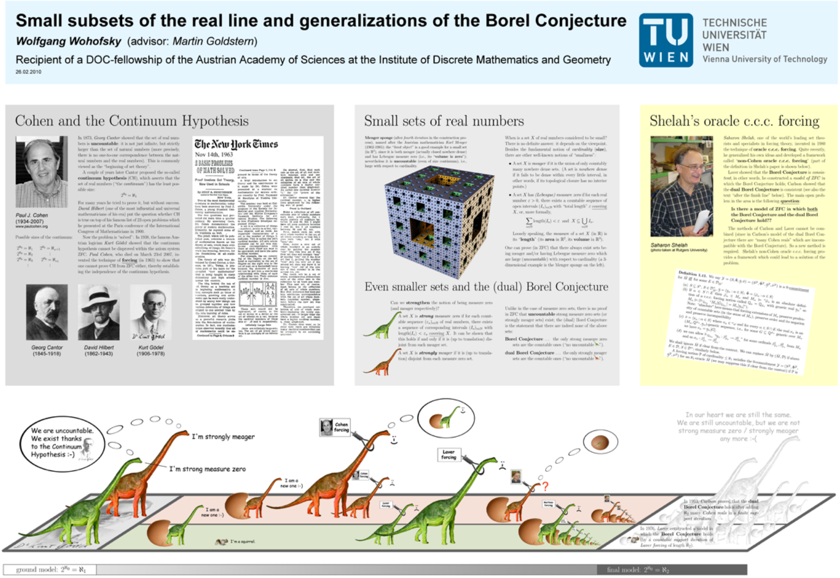
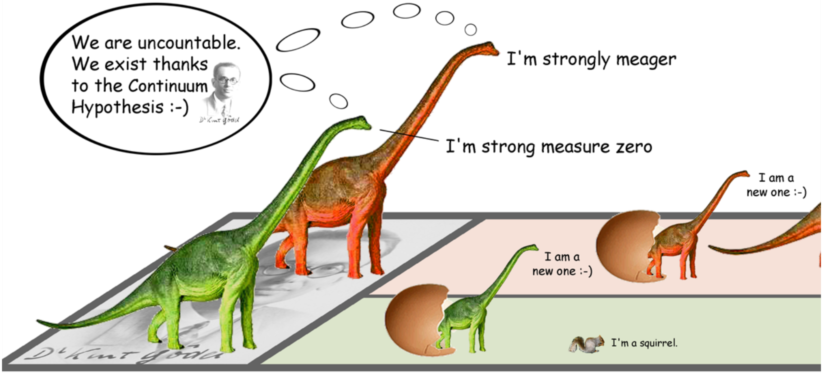

Small subsets of the real line and generalizations of the Borel Conjecture
Wolfgang Wohofsky (advisor: Martin Goldstern)
Recipient of a DOC-fellowship of the Austrian Academy of Sciences at the Institute of Discrete Mathematics and Geometry
26.02.2010
TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology
Cohen and the Continuum Hypothesis
Paul J. Cohen (1934-2007)
Georg Cantor (1845-1918)
David Hilbert (1862-1943)
Kurt Gödel (1906-1978)
The New York Times
Nov 14th, 1963
Small sets of real numbers
Even smaller sets and the (dual) Borel Conjecture
Shelah's oracle c.c.c. forcing
Saharon Shelah
We are uncountable. We exist thanks to the Continuum Hypothesis :-)
I'm strongly meager
I'm strong measure zero
Cohen forcing
Laver forcing
In our heart we are still the same. We are still uncountable, but we are not strong measure zero / strongly meager any more :-(
ground model: $2^{\aleph_0} = \aleph_1$
final model: $2^{\aleph_0} = \aleph_2$



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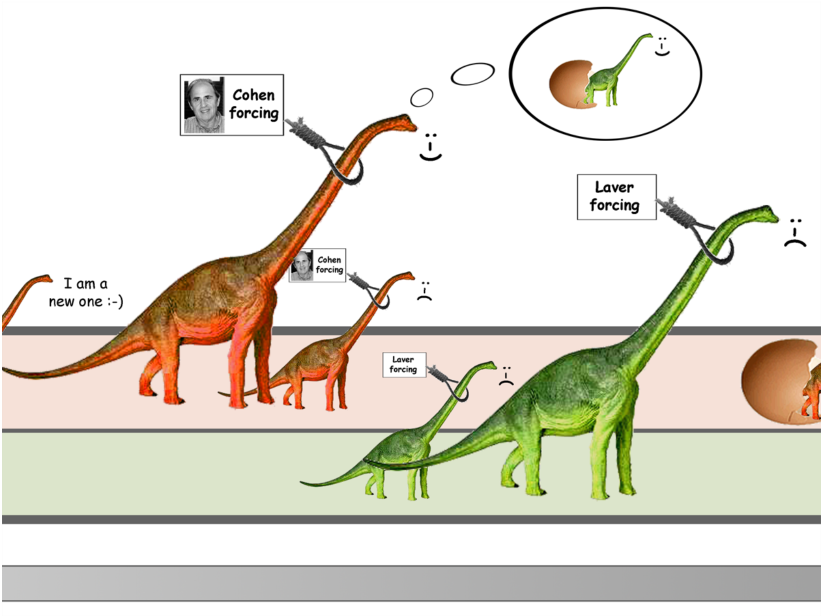
Cohen forcing
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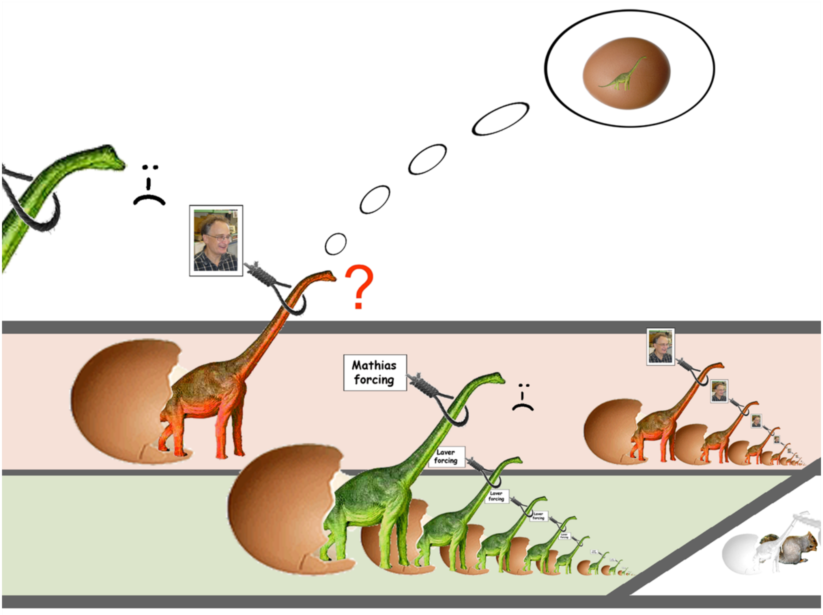
I am a new one :-)


?
Mathias
forcing
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The main theorem: Con(BC + dBC)

Theorem (Goldstern, Kellner, Shelah, W.; 2011+ε)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).

We force with $\mathbb{R} * \mathbb{P}_{\omega_2}$, where

- $\mathbb{R}$ is the preparatory forcing
  - a condition in $\mathbb{R}$ consists of
    - a (not quite transitive) countable model M
    - an iteration $(\bar{\mathbb{P}}^M, \bar{\mathbb{Q}}^M)$ in M
  - to get a stronger condition
    - "enlarge" the model
    - find an iteration into which the old one "canonically" embeds
  - σ-closed, $\aleph_2$-c.c.
- ...adding the "generic" forcing iteration $(\bar{\mathbb{P}}, \bar{\mathbb{Q}})$ with limit $\mathbb{P}_{\omega_2}$
  - each $\mathbb{Q}_\alpha$ is the union of the $\mathbb{Q}_\alpha^M$'s from the generic $G \subseteq \mathbb{R}$
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Theorem (Goldstern, Kellner, Shelah, W.; 2011+ε)

There is a model of ZFC in which both the Borel Conjecture and the dual Borel Conjecture hold, i.e., Con(BC + dBC).

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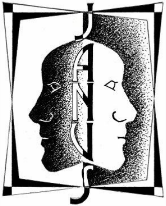

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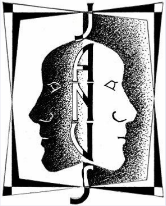

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Obtaining BC / dBC in the final model $V^{\mathbb{R}*\mathbb{P}_{\omega_2}}$

Theorem (Pawlikowski; 1993)

Let $X \subseteq 2^\omega$. Then X is strong measure zero if and only if $X + F$ is null for every *closed measure zero set* F.

To obtain the Borel Conjecture:

- kill uncountable strong measure zero sets X (by Ultralaver forcing)
  - witnessed by closed measure zero set F with $X + F$ positive
- prevent resurrection: show (down in M) that $X + F$ remains positive
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Another variant of the Borel Conjecture

- Small sets of real numbers
  - Real numbers, topology, measure, algebraic structure
  - Meager, measure zero, strong measure zero, Borel Conjecture (BC)
- Sets which can be translated away from an ideal $\mathcal{J}$
  - $\mathcal{J}^\star$, strongly meager, dual Borel Conjecture (dBC)
  - Main theorem: Con(BC + dBC)
- **Another variant of the Borel Conjecture**
  - Marczewski ideal s_0, ${s_0}^\star$, "Marczewski Borel Conjecture" (MBC)
  - "Sacks dense ideals", perfectly meager sets, Con(MBC)?

The "Borel Conjecture" for arbitrary ideals $\mathcal{J}$

Recall the definition of $\mathcal{J}^\star$ (for any $\mathcal{J} \subseteq \mathcal{P}(2^\omega)$):

$$\mathcal{J}^\star := \{Y \subseteq 2^\omega : Y + Z \neq 2^\omega \text{ for every set } Z \in \mathcal{J}\}.$$

From now on, assume that $\mathcal{J}$ is a translation-invariant σ-ideal. Then

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The $\mathcal{J}$-Borel Conjecture ($\mathcal{J}$-BC) the statement that there are **no** uncountable $\mathcal{J}$-shiftable sets, i.e., $\mathcal{J}^\star = [2^\omega]^{\leq\omega}$.

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A set $P \subseteq 2^\omega$ ($P \neq \emptyset$) is *perfect* iff it is closed and has no isolated points. It corresponds to the branches of a "perfect tree" in $2^{<\omega}$.

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Recall that both BC and dBC fail under CH.

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Replacing MA by PFA, we obtain the failure of MBC:

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PFA $\implies$ $\neg$MBC (actually ZFC $\vdash$ Con($\neg$MBC)).

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Sacks dense ideals (CH)

Unlike BC and dBC, the status of MBC under CH is unclear...

- Is MBC (i.e., ${s_0}^\star = [2^\omega]^{\leq\omega}$) consistent with CH?
- Or does CH even imply MBC?

I don't know, but in 2010 I obtained a partial result.

Definition (CH)

A collection $\mathcal{I} \subseteq \mathcal{P}(2^\omega)$ is a Sacks dense ideal iff

- $\mathcal{I}$ is a (non-trivial) translation-invariant σ-ideal
- $\mathcal{I}$ is dense in Sacks forcing, more explicitly, for each perfect $P \subseteq 2^\omega$, there is a perfect subset Q in the ideal, i.e., $Q \subseteq P$, $Q \in \mathcal{I}$

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Lemma (CH; from previous slide)

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In other words: $s_0{}^\star \subseteq \bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\}$.

Can we (consistently) find many Sacks dense ideals?

- the ideal $\mathcal{M}$ of meager sets is a Sacks dense ideal.
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Nevertheless we can "approximate $\mathcal{SN}$ from above" by Sacks dense ideals:

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A nice corollary (CH)

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Corollary (CH)

Assume CH. Then $s_0 \subsetneq s_0{}^{\star\star}$ (i.e., s_0 is NOT closed under **).

In contrast, CH implies both $\mathcal{M} = \mathcal{M}^{\star\star}$ and $\mathcal{N} = \mathcal{N}^{\star\star}$.

Proof.

- $s_0{}^\star \subseteq \mathcal{M}^\star$ (remember $\mathcal{SN} = \mathcal{M}^\star$)
- $s_0{}^{\star\star} \supseteq \mathcal{M}^{\star\star}$ and $\mathcal{M}^{\star\star} \supseteq \mathcal{M}$ ("Galois connection")
- $\mathcal{M}$ (hence $s_0{}^{\star\star}$) contains perfect sets, but s_0 does not.

□

A nice corollary (CH)

Theorem (CH; from previous slide)

Assume CH. $\bigcap\{\mathcal{I} : \mathcal{I} \text{ is a Sacks dense ideal}\} \subseteq \mathcal{SN}$.
Hence (by the Lemma) ${s_0}^\star \subseteq \mathcal{SN}$.

Corollary (CH)

Assume CH. Then $s_0 \subsetneqq {s_0}^{\star\star}$ (i.e., s_0 is NOT closed under **).

In contrast, CH implies both $\mathcal{M} = \mathcal{M}^{\star\star}$ and $\mathcal{N} = \mathcal{N}^{\star\star}$.

Proof.

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Thank you for your attention and enjoy the Winter School...