Classical and idealized MAD families

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- a ***P-ideal***, if for all countable $\{A_n : n \in \omega\} \subseteq \mathcal{I}$, there is a $B \in \mathcal{I}$ such that $A_n \subseteq^* B$ for $n \in \omega$ ($A \subseteq^* B \Leftrightarrow |A \setminus B| < \omega$),

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- ***Borel (analytic, meager, null, etc.)*** if $\mathcal{I} \subseteq \mathcal{P}(\omega) \simeq 2^\omega$ is Borel (analytic, meager, null, etc.) in the Cantor-space;
- ***tall*** if $\forall\ X \in [\omega]^\omega\ \mathcal{I} \cap [X]^\omega \neq \emptyset$.

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If $h : \omega \to (0, \infty)$ and $\sum_{n\in\omega} h(n) = \infty$, then the *summable ideal genarated by h*:

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Density ideals

Let $\vec{\mu} = \langle \mu_n : n \in \omega \rangle$ be a sequence of measures on ω with pairwise disjoint finite supports (P_n), and assume $\limsup_{n\to\infty} \mu_n(P_n) > 0$. Then the *density ideal associated to $\vec{\mu}$:*

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A function $\varphi : \mathcal{P}(\omega) \to [0, \infty]$ is a ***submeasure*** *on* ω if

(1) $\varphi(\emptyset) = 0$;

(2) $X \subseteq Y \subseteq \omega \Rightarrow \varphi(X) \leq \varphi(Y)$;

(3) $X, Y \subseteq \omega \Rightarrow \varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$;

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Remark

Lsc submeasures are σ-subadditive as well (that is, $\varphi(\bigcup_{n\in\omega} A_n) \leq \sum_{n\in\omega} \varphi(A_n)$ if $A_n \subseteq \omega$).

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Let $\mathcal{I}$ be an ideal on ω.

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Examples

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$\mathcal{I}_h = \mathrm{Fin}(\varphi_h) = \mathrm{Exh}(\varphi_h)$ where $\varphi_h(A) = \sum_{n \in A} h(n)$.

Remark (Farah)

There are F_σ P-ideals which are not summable.

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$\mathcal{Z}_{\vec{\mu}} = \mathrm{Exh}(\varphi_{\vec{\mu}})$ where $\varphi_{\vec{\mu}}(A) = \sup_{n \in \omega} \mu_n(A \cap P_n)$.

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$\mathfrak{a}(\mathcal{I}) > \omega$ for each F_σ ideal $\mathcal{I}$ but $\mathfrak{a}(\mathcal{Z}_{\vec{\mu}}) = \omega$ for tall density ideals.

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Proposition (F.-Soukup) – Lower and upper bounds for $\bar{\mathfrak{a}}(\mathcal{I})$

$\mathfrak{b} \leq \bar{\mathfrak{a}}(\mathcal{I})$ for each $F_{\sigma\delta}$ P-ideal $\mathcal{I}$ (but not for all F_σ ideals (Brendle)), and $\bar{\mathfrak{a}}(\mathcal{Z}_{\vec{\mu}}) \leq \mathfrak{a}$ for each density ideal $\mathcal{Z}_{\vec{\mu}}$.

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CH implies that there exist uncountable Cohen- and random-indestructible $\mathcal{I}$-MAD families for all F_σ ideals and $F_{\sigma\delta}$ P-ideals.

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Can any AD family be extended to a Cohen- (or random-) indestructible MAD family in a ccc forcing extension?

Motivation and a general question

Theorem (Fuchino, Geschke, Soukup)

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Idealized question

Assume $\mathcal{I}$ is an analytic ideal on ω, $\mathcal{A}$ is a $\mathcal{I}$-AD family, and let $\mathbb{F}$ be a forcing notion. Can $\mathcal{A}$ be extended to an $\mathbb{F}$-indestructible $\mathcal{I}$-MAD family in a ccc forcing extension?

$\mathbb{F}$-indestructible extensions for $\mathbb{F} \in V$

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The κ^+-stage iteration kills all possible $\dot{X}$ in the final model which could destroy our extended $\mathcal{I}$-AD family.

The Katětov (pre)order

Definition

If $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω (or on countable sets) then $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ iff there is an $F \in \omega^\omega$ such that $\forall\, A \in \mathcal{I}\; F^{-1}[A] \in \mathcal{J}$.

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The Katětov order is upward directed and $\mathfrak{c}^+$-downward directed (even on tall ideals). $\mathrm{Fin} = [\omega]^{<\omega}$ is a $\leq_{\mathrm{K}}$-minimal element, moreover $\mathcal{I} \not\leq_{\mathrm{K}} \mathrm{Fin}$ iff $\mathcal{I}$ is tall.

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Fact

Shoenfield's Absoluteness Theorem implies that $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ for Borel ideals is absolute between any pair of transitive models $M \subseteq N$ with $\omega_1^N \subseteq M$.

Characterizing forcing-indestructibility

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Clearly, $\mathrm{tr}(I)$ is an ideal in $\omega^{<\omega}$ (or on $2^{<\omega}$).

Characterizing forcing-indestructibility

For a set $A \subseteq \omega^{<\omega}$ (or $A \subseteq 2^{<\omega}$) its ***G_δ-closure*** is

$$G_\delta(A) = \{f \in \omega^\omega (\text{or } 2^\omega) : \exists^\infty\, n\; f \restriction n \in A\}.$$

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Theorem (Hrušák-Zapletal)

Let I be a σ-ideal on ω^ω (or on 2^ω) and assume that $\mathbb{P}_I = \mathrm{Borel}(\omega^\omega)/I$ is proper with the continuous reading of names (CRN). If $\mathcal{A}$ is a MAD family on ω, then the following are equivalent:

(1) There is a $B \in \mathbb{P}_I$ such that $B \Vdash$"$\mathcal{A}$ is not maximal".

(2) There is an $X \in \mathrm{tr}(I)^+$ such that $\mathcal{A} \leq_{\mathrm{K}} \mathrm{tr}(I) \restriction X$.

$\mathbb{B}$-indestructible extensions

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Corollary

Cohen-indestructibility $\Rightarrow$ Miller-indestructibility $\Rightarrow$ Sacks-indestructibility (of MAD families) so our theorem about $\mathbb{F}$-indestructible extensions of $\mathcal{I}$-AD families also works for the Miller- and Sacks-forcing if $\mathcal{I} = [\omega]^{<\omega}$.

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Assume $\mathcal{A}$ is an AD family. Then $\mathcal{A}$ can be extended to a random-indestructible MAD family in a ccc forcing extension.

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We will define an ω_1-stage finite support iteration of ccc forcing notions and extend $\mathcal{A}$ with one element at each stage by the following forcing notion: $(s, n, \mathcal{B}) \in \mathbb{Q}$ iff $s \subseteq n \in \omega$ and $\mathcal{B} \in [\mathcal{A}]^{<\omega}$,
$(s_0, n_0, \mathcal{B}_0) \leq (s_1, n_1, \mathcal{B}_1)$ iff

(a) $n_0 \geq n_1$, $s_0 \cap n_1 = s_1$ and $\mathcal{B}_0 \supseteq \mathcal{B}_1$;

(b) $(s_0 \setminus s_1) \cap \bigcup \mathcal{B}_1 = \emptyset$.

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$\mathbb{Q}$ is clearly σ-centered. Let $\dot{S}$ be the union of the first coordinates of conditions in the $\mathbb{Q}$-generic filter. Then $\Vdash_{\mathbb{Q}}$"$\mathcal{A} \cup \{\dot{S}\}$ is an AD family." Assume that $X = \{x_k : k \in \omega\} \in \mathcal{P}(2^{<\omega})$, $\varepsilon = \lambda(G_\delta(X)) > 0$, and $F : X \to \omega$ such that $\forall\, A \in \mathcal{A}\; F^{-1}[A] \in \mathrm{tr}(\mathcal{N}) \restriction X$.

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We claim that $V^{\mathbb{Q}} \models \lambda\big(G_\delta(F^{-1}[\dot{S}])\big) = \varepsilon$ so in $V^{\mathbb{Q}}$ F cannot show that $\mathcal{A} \cup \{\dot{S}\} \leq_{\mathrm{K}} \mathrm{tr}(\mathcal{N}) \restriction X$.

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If $p = (s^p, n^p, \mathcal{B}^p) \in \mathbb{Q}$ then

$$G_\delta(F^{-1}[\dot{S}]) = \bigcap_{k \in \omega} \left\{ f \in 2^\omega : m \geq k, F(x_m) \in \dot{S},\ \text{and } x_m \subseteq f \right\}.$$

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We have to show that the measure of the set after the intersection is at least ε so that the following sets are dense in $\mathbb{Q}$:

$$D_k^\delta = \{p \in \mathbb{Q} : \lambda\big(\{f \in 2^\omega : m \geq k, F(x_m) \in s^p\}\big) > \delta\}$$

where $\delta < \varepsilon$ and $k \in \omega$. It is followed by our assumption on F.

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The ω_1-stage iteration kills all possible Katětov-reduction of our family to $\mathrm{tr}(\mathcal{N}) \restriction X$ for some $X \in \mathrm{tr}(\mathcal{N})^+$.

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Problem 1 (maybe easy)

Can we characterize analytic ideals which has countable $\mathcal{I}$-MAD families?

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Problem 4

Can we characterize $\mathbb{P}_I$-indestructibility of $\mathcal{J}$-MAD families for proper $\mathbb{P}_I$'s with the CRN and F_σ ideals or $F_{\sigma\delta}$ P-ideals (or even for analytic ideals)?

Thank you for your attention!

(and please feel free to solve my questions☺)