Distributivity of Cohen forcing in larger universes

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General setting

Let $V \subseteq W$ be two transitive models of set theory with the same cardinals up to and including κ (κ regular). Let $P \in V$ be the Cohen forcing Add$(\kappa, 1)$ as defined in V.

Question. Is P still κ-distributive (non-collapsing) over W?

Clearly, the answer depends on the relationship between V and W. The question is interesting when $[\kappa]^{<\kappa}$ of W is *not* included in V, or when cofinalities change.

Product forcing: Example 1.

Let GCH hold in V and let κ be a regular cardinal in V. Let $Q = \text{Add}(\kappa, \lambda)$, where λ is any ordinal > 0, and $P = \text{Add}(\kappa^+, 1)$. Both P and Q are defined in V.

Claim 1 *P is still κ^+-distributive over V^Q (Easton's lemma).*

It follows that the preservation of distributivity does not depend simply on how many subsets of κ are missing from V.

Product forcing: Example 2.

(Shelah) There are two proper forcing notions P and Q, where P may be taken to be Add($\omega_1, 1$) such that $Q \times P$ collapses ω_1.

In particular,

Claim 2 *P is not ω_1-distributive over V^Q.*

Large cardinals

Let M be a transitive class. We say that a non-trivial (not an identity) $j : V \to M$ is elementary if

$$\varphi(x_1, \ldots, x_n) \to (\varphi(j(x_1), \ldots, j(x_n)))^M$$

is true for every formula φ and $x_1, \ldots, x_n$ in V.

Kunen's result implies that $M \neq V$.

V has its isomorphic copy as a non-transitive proper subclass of M, denoted as $j[V]$. The unique transitive collapse of $j[V]$ is V.

If there exists $j : V \to M$ with critical point κ, then κ is called a *measurable cardinal*.

Product forcing: Example 3. Let κ be a measurable cardinal and let $Q = \text{Prk}(\kappa)$ be the plain Prikry forcing which adds an ω-cofinal sequence through κ, without adding new bounded subsets of $H(\kappa)$. Let $P = \text{Add}(\kappa, 1)$. Both Q and P are defined in V. Then

Claim 3 *P is not κ-distributive over V^Q, in fact $Q \times P$ collapses all cardinals in the interval $(\omega, \kappa]$.*

Note that in this case $Q \times P$ is isomorphic to $Q * P$.

Example 4: Elementary embeddings.

Let $j : V \to M$ be an elementary embedding, and $P = \mathsf{Add}(\lambda, 1)$ for some V-regular cardinal λ. P is defined in M.

Question. When is P λ-distributive over V?

Note that in this case V is not a generic extension of the smaller model M, and hence new methods for answering the question above seem to be necessary.

Why is the above question interesting?

We say that $j : V \to M$ with critical point κ is κ^{++}-correct, if:

(i) M is closed under κ-sequences in V,
(ii) $(\kappa^{++})^M = \kappa^{++}$.

Existence of such an embedding follows, and is in fact equivalent in terms of consistency, to an existence of κ with $o(\kappa) = \kappa^{++}$.

Question. Assume GCH. Let $j : V \to M$ be κ^{++}-correct embedding. Let $P = \mathrm{Add}(\kappa^{++}, 1)^M$. Is P κ^{++}-distributive over V?

Lemma 4 (Key lemma) *Assume GCH and* $j : V \to M$ *be a* κ^{++}*-correct embedding. Then there is a forcing* $\mathbb{P}$ *such that if* G *is* $\mathbb{P}$*-generic over* V*, then there is a* κ^{++}*-correct embedding* $j^* : V[G] \to M^*$ *such that*

$$\mathrm{Add}(\kappa^{++}, 1)^{M^*} \text{ is } \kappa^{++}\text{-distributive over } V[G].$$

This lemma is crucial in the proof of:

Theorem 5 (Sy Friedman, H., '11) *(A simple version) The following are equiconsistent:*

(i) There is κ *such that* $o(\kappa) = \kappa^{++}$.

(ii) There is κ *such that* κ *is measurable,* $2^\kappa = \kappa^{++}$ *and* $2^\alpha = \alpha^{++}$ *for every regular cardinal* $\alpha < \kappa$.

Why is this theorem interesting?

The continuum function

Consider the function from cardinals to cardinals such that

$$\kappa \mapsto 2^\kappa.$$

We call this the *continuum function*. The continuum function at κ depends on the continuum function on cardinals $< \kappa$ if κ is:

(i) a singular (strong limit) cardinal of uncountable cofinality,
(ii) a large cardinal (such as a measurable cardinal).

If κ is a regular (not large) cardinal, then 2^κ does not depend on $\alpha < \kappa$ (Easton).

Ad (i). (Silver) Suppose κ is a strong limit singular cardinal of uncountable cofinality. If $2^\alpha = \alpha^+$ for stationary many regular $\alpha < \kappa$, then $2^\kappa = \kappa^+$.

Ad (ii). Suppose κ is a measurable cardinal. If the set of all regular cardinals $\alpha < \kappa$ such that $2^\alpha = \alpha^+$ is the set of all regulars in a club in κ, then $2^\kappa = \kappa^+$.

Thus there is a delicate connection between strong limit cardinals of uncountable cofinality, and large cardinals.

(Gitik). The following are equiconsistent:

(i) There exists κ with $o(\kappa) = \kappa^{++}$.

(ii) There exists a measurable cardinal κ such that $2^\kappa = \kappa^{++}$.

Compare with

(ii*) [F-H] There is κ such that κ is measurable, $2^\kappa = \kappa^{++}$ and $2^\alpha = \alpha^{++}$ for every regular cardinal $\alpha < \kappa$.

Key Lemma is one of the main ingredients in proving (ii*) from (i).

Proof of theorem

Key lemma. Assume GCH and $j : V \to M$ be a κ^{++}-correct embedding. Then there is a forcing $\mathbb{P}$ such that if G is $\mathbb{P}$-generic over V, then there is a κ^{++}-correct embedding $j^* : V[G] \to M^*$ such that

Add$(\kappa^{++}, 1)^{M^*}$ is κ^{++}-distributive over $V[G]$.

Sketch of proof of Key Lemma. The proof is inspired by an idea by U. Abraham.

Set $\mathbb{P}$ be be a reverse Easton iteration of Add(α^+, α^{++}) for each inaccessible cardinal $\alpha \leq \kappa$. Let G be $\mathbb{P}$-generic, and let us write $G = G_\kappa * g$ where g is Add$(\kappa^+, \kappa^{++})^{V[G_\kappa]}$-generic over $V[G_\kappa]$. By standard arguments j lifts to $j^* : V[G] \to M[j^*(G)] = M^*$.

We argue that $P = \text{Add}(\kappa^{++}, 1)^{M[G]} = \text{Add}(\kappa^{++}, 1)^{M^*}$ is still κ^{++}-distributive over $V[G]$.

Let $p \Vdash \dot{f} : \kappa^+ \to$ On hold in $V[G]$, for $p \in P$. Let N be an elementary substructure of some $H(\theta)^{V[G]}$ of size κ^+, closed under κ-sequences, and transitive below κ^{++}, containing $P, p, \dot{f}$.

N is not in $M[G]$, but look at $P \cap N$. Let $\bar{N}$ be the transitive collapse of N by π. Then $\pi(P) = P \cap N$, $\pi(P)$ is in $M[G]$ (because P is definable in $H(\kappa^{++})$ of $M[G]$, which can be viewed as $L_{\kappa^{++}}[B]$ for some $B \subseteq \kappa^{++}$ in $M[G]$, and so $\pi(P)$ is in $L_{N \cap \kappa^{++}}[B \cap N \cap \kappa^{++}] \subseteq M[G]$).

Now, we show that all dense open subsets of $\bar{N}$ in $\bar{N}$ can be met by a decreasing κ^+ sequence $\langle p_i \,|\, i < \kappa^+ \rangle$ of condition in $\pi(P)$, the sequence being defined in $M[G]$. Then $q = \lim_i p_i$ is in $M[G]$ and decides $\dot{f}$.

Note that $\bar{N}$ is not in $M[G]$, so how can we obtain such a $\bar{N}$-generic sequence in $M[G]$?

We use the "guiding generic" g. By a density argument, the guiding generic g makes sure that we hit all dense open sets in $\bar{N}$.

In more detail, choose $\gamma < \kappa^{++}$ such that $V[G_\kappa * g \restriction \gamma]$ and $M[G_\kappa * g \restriction \gamma]$ contain all necessary parameters:

–$V[G_\kappa * g \restriction \gamma]$ contains $\bar{N}$,
–$M[G_\kappa * g \restriction \gamma]$ contains $\pi(P)$ and an enumeration $\langle p'_i \mid i < \kappa^+ \rangle$ of $\pi(P)$.

This is possible by κ^{++}-cc of Add(κ^+, κ^{++}).

Define $\langle p_i \,|\, i < \kappa^+ \rangle$ in $M[G_\kappa * g \restriction \gamma][g(\gamma)]$:

$$p_{i+1} = \begin{cases} p'_{g(\gamma)(i)} & \text{if } p'_{g(\gamma)(i)} \text{ extends } p_i, \\ p_i & \text{otherwise.} \end{cases}$$

Finally, in $V[G_\kappa * g \restriction \gamma]$ one argues that if $D \in \bar{N}$ is dense in $\pi(P)$, then the following set is dense in $\text{Add}(\kappa^+, 1)$:

$$\bar{D} = \{q \,|\, q \Vdash \text{``}\exists i < \kappa^+, p_i \in D\text{''}\}.$$

Proof, cont'd.

Assume GCH, and let $j : V \to M$ be κ^{++}-correct. Let $P = \mathrm{Add}(\kappa^{++}, \kappa^{+4})^M$.

Claim 6 *P usually collapses κ^{++} to κ^{+} if forced over V.*

Proof. Use an extender ultrapower representation which gives that $(\kappa^{+4})^M$ has cof κ^{+} in V.

Lemma 7 (2nd Key Lemma) *If* h *is* $\text{Add}(\kappa^{++}, 1)^{M[G]}$*-generic over* $V[G]$*, then one can "stretch"* h *into some some* h' *such that* h' *is* $\text{Add}(\kappa^{++}, \kappa^{+4})^{M[G]}$*-generic over* $M[G]$*.*

Proof. Find a "locally correct" bijection $\pi : (\kappa^{+4})^M \to \kappa^{++}$ such that if $X \subseteq (\kappa^{+4})^M$ in M has size $\leq \kappa^{++}$ in M, then $\pi \restriction X$ is in M.

Some generalizations:

- (A vague version of theorem) The following are equiconsistent:

  (i) There is κ such that $o(\kappa) = \kappa^{++}$.

  (ii) There is κ such that κ is measurable, $2^\kappa = \kappa^{++}$ and the continuum function on regular cardinals below κ is anything one wants (consistent with the provable limitations).

- The above generalizes to all $n < \omega$, with $o(\kappa) = \kappa^{+n}$.

  The case of $o(\kappa) = \kappa^{+\beta}$ for $\beta \geq \omega$ is more involved, but we expect no difficulties.

- **Question.** Is there a κ^{++}-correct $j : V \to M$ such that $\mathrm{Add}(\kappa^{++}, 1)^M$ is not κ^{++}-distributive over V?

- Classification of embeddings by preservation of combinatorial properties of forcing notions.