Playing With Projections: Ultrafilters, Mathias Forcing and Cardinal Invariants with Closed Subspaces of l^2

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Hilbert Space Projections and Subspaces

Definition

A *Hilbert space* is a (real or complex) vector space H together with a complete inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}(= \mathbb{R} \text{ or } \mathbb{C})$.

- $H = l^2 = \{(x_n) \subseteq \mathbb{F} : \sum |x_n|^2 < \infty\}$ and $\langle (x_n), (y_n) \rangle = \sum x_n \overline{y_n}$.

Definition

A *projection* $P_V \in \mathcal{P}(H)$ onto $V \in \overline{\mathcal{V}}(H)$ is a linear operator such that

$$\forall x \in H \quad Px \in V \quad \text{and} \quad x - Px \perp V.$$

- For $U, V \in \overline{\mathcal{V}}(H)$,

$$U \subseteq V \Leftrightarrow P_U \leq P_V \Leftrightarrow P_U P_V = P_U = P_V P_U.$$

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Modulo Compact Operators

- $\mathcal{K}(H) = \{T \in \mathcal{B}(H) : \overline{T[B_1(H)]} \text{ is compact}\}$.

  $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$.

  $\pi : \mathcal{B}(H) \rightarrow \mathcal{C}(H)$ is the canonical homomorphism.

- $U \leq^* V \Leftrightarrow P_U \leq^* P_V \Leftrightarrow \pi(P_U P_V) = \pi(P_U) = \pi(P_V P_U)$.

- Basis $(e_n)_{n\in\omega} \subseteq H$. Canonical Embedding $\mathscr{P}(\omega) \mapsto \overline{\mathcal{V}}(H)$,

$$A \mapsto V_A = \overline{\text{span}}\{e_n : n \in A\}.$$

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Basic Order Properties

$\mathcal{P}(H)$ is *not* a lattice.

Theorem

$$\sup(\sigma_e(PQP) \cap [0,1)) < 1.$$

$\Leftrightarrow \exists P \wedge^* Q.$

$\Leftrightarrow \exists P \vee^* Q.$

$\Leftrightarrow \exists^*\text{-max } V \subseteq \mathcal{R}(P) + \mathcal{R}(Q).$

Theorem

Arbitrary $(V_n) \subseteq \overline{\mathcal{V}}(H)$ has $\leq^*$-equivalent decreasing $(U_n) \subseteq \overline{\mathcal{V}}(H)$.

Corollary

No non-trivial finite or countable gaps in $\overline{\mathcal{V}}(H)$.

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1. *prefilter base* (on $\mathbb{P}$) if $\forall \mathcal{G} \in [\mathcal{F}]^{<\omega} \exists p \in \mathbb{P} \forall g \in G (p \leq g)$.
2. *filter base* if $\mathcal{F}$ is a prefilter base on itself.
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4. *ultra[pre]filter* (on $\mathbb{P}$) if $\mathcal{F}$ a [pre]filter base maximal in $\mathbb{P}$.

- A filter $\mathcal{F}$ is an ultraprefilter $\Leftrightarrow \forall p \in \mathbb{P} (p \in \mathcal{F} \vee p \top q \in \mathcal{F})$.
- If $\mathbb{P}$ is a lower semilattice then prefilters extend to filters.
- Then ultrafilters and ultraprefilters coincide.

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P-Points

$\mathcal{U} \subseteq [\omega]^\omega$ is an ultrafilter

- $\mathcal{U}$ P-point $\Rightarrow V_\mathcal{U} = \{V_U : U \in \mathcal{U}\}$ ultraprefilter base.
- Otherwise $V_\mathcal{U}$ not in an ultraprefilter filter.

$V_\mathcal{U}$ not an ultrafilter example

Take IP $(I_n) \subseteq \omega$, $|I_n| \to \infty$, $v_n = \sum_{k \in I_n} e_k$ and $V_{(I_n)} = \overline{\mathrm{span}}(v_n)$.
Take ultrafilter $\mathcal{U}$ on ω s.t. $\forall U \in \mathcal{U} \limsup |U \cap I_n|/|I_n| > 0$.
Note: $\forall U \in \mathcal{U}(V_U \not\leq^* V_{(I_n)}^\perp \wedge \dim(V_U \cap V_{(I_n)}^\perp) = \infty)$.
$\therefore \{V_U \cap V : U \in \mathcal{U}\}$ extends upwards-closre of $V_\mathcal{U}$.

Question

Can $V_\mathcal{U}$ be an ultrafilter for non-P-point $\mathcal{U}$?

Theorem

All ultraprefilter filters on $\mathcal{P}(H)$ are σ-closed (P-points).

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Proposition

Every $\mathcal{P}(H)$-generic is an ultraprefilter Q-point.

Questions

Complete combinatorics? Other special ultrafilters? Rudin-Kiesler?
Are $(\overline{\mathcal{V}}(H), \leq^*)$ and $(\overline{\mathcal{V}}_\infty(H), \subseteq)$ forcing equivalent?
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$X \times \mathbb{P}$ is *Mathias-like* if

1. $(x, p) \leq (y, q) \Rightarrow p \leq q$, and
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If $X \times \mathbb{P}$ is Mathias-like it is densely embeddable in $\mathbb{P} * (X \times \dot{G})$.

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If $\omega \times \mathbb{P}$ is Mathias-like and $\mathbb{P}$ is proper so is $X \times \mathbb{P}$.

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$X \times \mathbb{P}$ is *Mathias-like* if

1. $(x, p) \leq (y, q) \Rightarrow p \leq q$, and
2. $p \leq q \Rightarrow (x, p) \leq (x, q)$.

Proposition

If $X \times \mathbb{P}$ is Mathias-like it is densely embeddable in $\mathbb{P} * (X \times \dot{G})$.

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If $\omega \times \mathbb{P}$ is Mathias-like and $\mathbb{P}$ is proper so is $X \times \mathbb{P}$.

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$$[\mathcal{P}_\infty(H)]^{<\omega}_{\min} = \{(\mathcal{P}, P) : P \in \mathcal{P} \in [\mathcal{P}_\infty(H)]^{<\omega} \wedge \forall Q \in \mathcal{P}(P \leq^* Q)\}$$

$$(\mathcal{P}, P) \leq (\mathcal{Q}, Q) \Leftrightarrow \mathcal{P} \supseteq \mathcal{Q}.$$

For dense $(v_n) \subseteq H$, $\mathcal{V}^{<\infty}_{(v_n)} = \{\mathrm{span}_{n\in F}(v_n) : F \in [\omega]^{<\omega}\}$.

$$\mathbb{M}^* = \mathcal{V}^{<\infty}_{(v_n)} \times \omega \times [\mathcal{P}_\infty(H)]^{<\omega}_{\min}.$$

$(V, n, \mathcal{P}, P) \leq (W, m, \mathcal{Q}, Q) \Leftrightarrow$
$V \supseteq W \wedge \mathcal{P} \supseteq \mathcal{Q} \wedge \forall R \in \mathcal{Q}(||R|_{V\cap W^\perp}|| + 1/n \leq 1/m)$,

Transitivity: if $||R|_{V\cap W^\perp}|| + 1/n \leq 1/m$ and $||R|_{W\cap X^\perp}|| + 1/m \leq 1/l$,

$$||R|_{V\cap X^\perp}||+1/n \leq ||R|_{V\cap W^\perp}||+||R|_{W\cap X^\perp}||+1/n \leq ||R|_{W\cap X^\perp}||+1/m \leq 1/l.$$

Splitting

- $\mathscr{P}(\omega)/\text{Fin}$ cardinal invariants have $(\geq)2$ analogs $\because$

$$\begin{array}{ccc} P \wedge^* Q^\perp \neq 0 & \Rightarrow & P \not\leq^* Q \\ \Updownarrow & \not\Leftarrow & \Updownarrow \\ ||\pi(PQ^\perp)|| = 1 & & ||\pi(PQ^\perp)|| > 0 \end{array}$$

- P strongly splits $Q \Leftrightarrow P \wedge^* Q \neq 0 \neq P^\perp \wedge^* Q$.

  P weakly splits $Q \Leftrightarrow P \wedge^* Q \neq 0$ and $Q \not\leq^* P$.

  $\mathfrak{s}^\perp = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}(H)$ is a strongly splitting family$\}$.

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Block Subspaces

- V is a block subspace of $H = \mathrm{span}(e_n)$ means
  $\exists$ IP (I_n) and $\exists (v_n) \subseteq l^2$ s.t.
  $V = \mathrm{span}(v_n)$ and $\forall n\ v_n \in \mathrm{span}\{e_k : k \in I_n\}$.
- Block subspaces are $\leq^*$-dense.
  Given inf dim $V \subseteq H$ recursively pick unit vectors $(v_n) \subseteq V$

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\begin{aligned}
v_0 &= (0, \frac{1}{5}, \frac{3}{4}, \frac{1}{2}, \frac{1}{10}, \ldots) \quad \text{(arbitrary)} \\
v_1 &= (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{16}}, \ldots) \in V \cap l^{2\perp}_{k_0},\ k_0 >> 0 \\
v_2 &= (0, 0, 0, 0 \ldots, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \ldots) \in V \cap l^{2\perp}_{k_1},\ k_1 >> k_0 \\
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  $V \supseteq \mathrm{span}(v_n) =^*$ block subspace. $\square$

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Interval Paritions

- Consequence: card invs on $\mathcal{P}(\mathcal{C}(H))$ often related to IP card invs.
- Eg. $A \subseteq \omega$ splits IP (I_n) $\Leftrightarrow$ $\exists^\infty n\ I_n \subseteq A$ and $\exists^\infty n\ I_n \subseteq \omega \setminus A$.

  $\mathfrak{s}^{\mathrm{IP}} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{P}(\omega)$ is an IP splitting family$\}$.

  $\mathcal{A} \subseteq \mathscr{P}(\omega)$ IP splitting $\Rightarrow (P_A)_{A\in\mathcal{A}}$ strongly splitting.

  $\Rightarrow \mathfrak{s}^\perp \leq \mathfrak{s}^{\mathrm{IP}}$.
- $\mathfrak{s}^{\mathrm{IP}} = \max(\mathfrak{s}, \mathfrak{b})$ (A. Kamburelis, B. Weglorz (1995)).

$$\mathfrak{p} \leq \mathfrak{s}^* \leq \mathfrak{s}^\perp \leq \max(\mathfrak{s}, \mathfrak{b})$$

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- Given IP (I_n) set $v_n = \sum_{k \in I_n} e_k$ and $V_{(I_n)} = \text{span}(v_n)$.

  If $A \subseteq \omega$ s.t. $|A \cap I_n|/|I_n| \to 1$ then $V_{(I_n)} \subseteq^* V_A$ (Wofsey).

  MA $\Rightarrow \exists$ tower $(A_\xi) \subseteq [\omega]^\omega$ s.t. $\forall \xi |A_\xi \cap I_n|/|I_n| \to 1$
  $\Rightarrow \forall \xi V_{(I_n)} \subseteq^* V_{A_\xi}$, i.e. (V_{A_ξ}) is not a tower.
- $\mathfrak{t} = \mathfrak{b} \Rightarrow \exists$ tower $(A_\xi) \subseteq [\omega]^\omega$ s.t. (V_{A_ξ}) is a tower.
- $\{A \subseteq \omega : V_A \leq^* V\}$ is an analytic p-ideal (Steprans).

  Consistently no towers in $\mathcal{I}^*$ $\forall$ such ideals $\mathcal{I}$ (Brendle).

  Corollary: Consistently all towers in $[\omega]^\omega$ remain towers.
- $\mathfrak{t}^* = \mathfrak{t}^\perp . = \mathfrak{t}$? $\mathfrak{t}^* \geq \mathfrak{p}$ (Wofsey+Bell) so if $\mathfrak{t}^* < \mathfrak{t}$ then $\mathfrak{p} < \mathfrak{t}$.

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Definition (*-Orthogonality)

For $P, Q \in \mathcal{P}(H)$, $\quad P \perp^* Q \Leftrightarrow \pi(PQ) = 0$.

- $\mathfrak{a}^* = \mathfrak{a}(\perp^*) = \min\{|\mathcal{P}| \geq \aleph_0 : \mathcal{P} \text{ max s.t. } \forall P, Q \in \mathcal{P}(P \perp^* Q)\}$.
- $\mathfrak{b} \leq \mathfrak{a}^*$ (Brendle).
- (Wofsey) Consistently: $\mathfrak{a}^* = \mathfrak{a} = \aleph_1 < \mathfrak{c}$ (finite conditions), $\mathfrak{a}^* = \mathfrak{a} = \mathfrak{c} > \aleph_1$ (MA).

Definition (*-Incompatibility)

For $P, Q \in \mathcal{P}(H)$, $\quad P \top^* Q \Leftrightarrow ||\pi(PQ)|| < 1$.

- $\mathfrak{a}(\top^*)$ = min size of max $\leq^*$-antichain.
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