Structural Ramsey theory and topological dynamics III

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February 2011

- From previous lecture: When $\mathbb{F}$ countable ordered ultrahomogeneous structure, $\mathrm{Aut}(\mathbb{F})$ is extremely amenable iff $\mathbb{F}$ has the Ramsey property.
- Today: What if $\mathrm{Aut}(\mathbb{F})$ is not extremely amenable?

Part V

Universal minimal flows

G-flows

Definition

Let G be a Hausdorff topological group.
A G-flow is a continuous action of G on a compact Hausdorff space X.
Notation: $G \curvearrowright X$.
$G \curvearrowright X$ is minimal when every $x \in X$ has dense orbit in X:

$$\forall x \in X \quad \overline{G \cdot x} = X$$

$G \curvearrowright X$ is universal when:

$$\forall G \curvearrowright Y \text{ minimal}, \quad \exists \pi : X \longrightarrow Y \text{ continuous, onto, and so that}$$
$$\forall g \in G \quad \forall x \in X \quad \pi(g \cdot x) = g \cdot \pi(x).$$

"Every minimal G-flow is a continuous image of $G \curvearrowright X$."

Universal minimal flow

Theorem (Folklore)

Let G be a Hausdorff topological group.
Then there is a unique G-flow that is both minimal and universal.
Notation: $G \curvearrowright M(G)$.

General question: Describe $G \curvearrowright M(G)$ explicitly when G is a "concrete" group.

Remark

- *$M(G)$ may not be metrizable (E.g. G discrete)*
- *G is extremely amenable iff $M(G)$ is a singleton.*

Proof.

If G is extremely amenable, then its action on $M(G)$ has a fixed point. by minimality, $M(G)$ is a singleton.
If $M(G)$ is a singleton, and G acts on a compact space, then any minimal subflow is a singleton, hence a fixed point. □

The first non-trivial metrizable universal minimal flow

Theorem (Pestov, 98)

$Homeo_+(\mathbb{S}^1) \curvearrowright M(Homeo_+(\mathbb{S}^1))$ is the natural action $Homeo_+(\mathbb{S}^1) \curvearrowright \mathbb{S}^1$.

Proof.

Fix $x \in \mathbb{S}^1$, $H := Stab(x)$.
Then $H \cong Homeo_+([0,1])$, extremely amenable.
Write $G = Homeo_+(\mathbb{S}^1)$, and let $G \curvearrowright X$ be minimal.
It induces $H \curvearrowright X$, so find $x_0 \in X$, H-fixed.
Let $\pi : G \longrightarrow X, g \mapsto gx_0$.
Clearly, if $g^{-1}h \in H$, then $\pi(g) = \pi(h)$.
So really, $\pi : G/H \longrightarrow X$.
Note: it is G-equivariant.
Check that $G/H \cong \mathbb{S}^1$, and that $G \curvearrowright G/H$ is the natural action $G \curvearrowright \mathbb{S}^1$.
Finally, π onto by minimality of X.

□

Applying Pestov's quotient method

Let $\mathbb{F}$ be countable, ultrahomogeneous.
Assume $\mathbb{F}^* = (\mathbb{F}, R_1^* \dots R_m^*)$ relational expansion of $\mathbb{F}$ with Ramsey property.
Then we can construct a universal $\mathrm{Aut}(\mathbb{F})$-flow as follows:
Write $G := \mathrm{Aut}(\mathbb{F})$, $G^* = \mathrm{Aut}(\mathbb{F}^*)$.
Let $G \curvearrowright X$ be minimal.
It induces $G^* \curvearrowright X$.
By Ramsey property for $\mathbb{F}^*$, G^* is extremely amenable, so find $x_0 \in X$, G^*-fixed.
Let $\pi : G \longrightarrow X$, $g \mapsto gx_0$.
Then $\pi(g)$ depends only on $[g] \in G/G^*$, and really, $\pi : G/G^* \longrightarrow X$.

$\pi : G/G^* \longrightarrow X$.

Note that for $g, h \in G$:

$$\begin{aligned} g^{-1}h \in G^* &\text{ iff } \forall i \leq m \ \ \forall \bar{x} \in \mathbb{F}^{a(i)} \ \ R_i^*(g^{-1}h\bar{x}) \Leftrightarrow R_i^*(\bar{x}) \\ &\text{ iff } \forall i \leq m \ \ \forall \bar{x} \in \mathbb{F}^{a(i)} \ \ R_i^*(g^{-1}\bar{x}) \Leftrightarrow R_i^*(h^{-1}\bar{x}) \\ &\text{ iff } \forall i \leq m \ \ gR_i^* = hR_i^* \text{ (logic action)} \end{aligned}$$

Therefore: $G/G^* = G \cdot (R_1^* \ldots R_m^*) \subset \prod_{i=1}^m 2^{\mathbb{F}^{a(i)}}$.

And so $\pi : G \cdot (R_1^* \ldots R_m^*) \longrightarrow X$

Thus, if π is "uniformly continuous", it extends to

$$\hat{\pi} : \overline{G \cdot (R_1^* \ldots R_m^*)} \longrightarrow X$$

Note that $\hat{\pi}$ is G-equivariant, and onto by minimality of X.

This proves that $G \curvearrowright \overline{G \cdot (R_1^* \ldots R_m^*)}$ is universal...

...And so any minimal subflow of $G \curvearrowright \overline{G \cdot (R_1^* \ldots R_m^*)}$ is universal and minimal, and hence is the universal minimal flow.

Remark

To have uniform continuity of π, the projection of the right-invariant metric of G onto G/G^* is the relevant one: $d_R(g, h) = d_L(g^{-1}, h^{-1})$.

Minimality of $X^* := \overline{G \cdot (R_1^* \dots R_m^*)}$

Definition

Say that $\mathrm{Age}(\mathbb{F}^*)$ *has the expansion property over* $\mathrm{Age}(\mathbb{F})$ *when*

$$\forall A \in \mathrm{Age}(\mathbb{F}),\ \exists B \in \mathrm{Age}(\mathbb{F})\ \forall A^*, B^* \text{ expansions of } A, B \text{ in } \mathrm{Age}(\mathbb{F}^*),\\ A^* \hookrightarrow B^*.$$

Theorem (KPT, 05)

$\mathrm{Aut}(\mathbb{F}) \curvearrowright X^*$ *minimal iff* $\mathrm{Age}(\mathbb{F}^*)$ *has expansion property over* $\mathrm{Age}(\mathbb{F})$.

Theorem (KPT, 05)

Let $\mathbb{F}$ *be countable, ultrahomogeneous.*
Assume $\mathbb{F}^* = (\mathbb{F}, R_1^* \dots R_m^*)$ *relational expansion of* $\mathbb{F}$. *TFAE:*

1. $\mathbb{F}^*$ *has the Ramsey property and* $\mathrm{Age}(\mathbb{F}^*)$ *has the expansion property over* $\mathrm{Age}(\mathbb{F})$.
2. $\mathrm{Aut}(\mathbb{F}) \curvearrowright M(\mathrm{Aut}(\mathbb{F})) = \mathrm{Aut}(\mathbb{F}) \curvearrowright X^* \subset \prod_{i=1}^m 2^{\mathbb{F}^{a(i)}}$ *(logic action).*

Strategy to find universal minimal flows

- Choose your favorite countable ultrahomogeneous structure $\mathbb{F}$.
- Consider its class $\mathrm{Age}(\mathbb{F})$ of finite substructures.
- Try to enrich $\mathrm{Age}(\mathbb{F})$ with finitely many relations (among which a linear ordering) to obtain a class $\mathcal{K}^*$ such that
  - $\mathcal{K}^*$ is a Fraïssé class with the Ramsey property.
  - $\mathcal{K}^*$ has the expansion property over $\mathrm{Age}(\mathbb{F})$.
- Express the limit of $\mathcal{K}^*$ as $(\mathbb{F}, R_1^* \ldots R_m^*)$.
- Describe the action $\mathrm{Aut}(\mathbb{F}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{F}) \cdot (R_1^* \ldots R_m^*)}$.
- Rk1: The original article deals only with $m = 1$, $R_1^* =<$, but generalizes easily to finite relational expansions.
- Rk2: All universal minimal flows obtained that way are metrizable.

Graphs

- $\mathcal{G}$ finite graphs:

Theorem (Nešetřil-Rödl, 77)

Let $\mathcal{G}^<$ be the class of all finite ordered graphs.
Then $\mathcal{G}^<$ has the Ramsey and the expansion property over $\mathcal{G}$.

Corollary

$\mathrm{Aut}(\mathcal{R}) \curvearrowright M(\mathrm{Aut}(\mathcal{R}))$ *is* $\mathrm{Aut}(\mathcal{R}) \curvearrowright LO(\mathcal{R})$.

- $\mathcal{H}_n$ finite K_n-free graphs:

Theorem (Nešetřil-Rödl, 77)

Let $\mathcal{H}_n^<$ be the class of all finite ordered K_n-free graphs.
Then $\mathcal{H}_n^<$ has the Ramsey and the expansion property over $\mathcal{H}_n$.

Corollary

$\mathrm{Aut}(H_n) \curvearrowright M(\mathrm{Aut}(H_n))$ *is* $\mathrm{Aut}(H_n) \curvearrowright LO(H_n)$.

Partial orders

• $\mathcal{P}$ finite partial orders:

Definition

Let $P \in \mathcal{P}$. A linear order on P is compatible when it extends $<^P$.

Theorem (Nešetřil, 05)

Let $\mathcal{P}^{e<}$ be the class of all finite compatibly ordered partial orders. Then $\mathcal{P}^{e<}$ has the Ramsey and the expansion property over $\mathcal{P}$.

Corollary

Let $eLO(\mathbb{P})$ be the class of all compatible linear orders on $\mathbb{P}$. Then $\mathrm{Aut}(\mathbb{P}) \curvearrowright M(\mathrm{Aut}(\mathbb{P}))$ is $\mathrm{Aut}(\mathbb{P}) \curvearrowright eLO(\mathbb{P})$.

Vector spaces

• $\mathcal{V}_F$ finite vector spaces, F finite field.

Definition

Let $V \in \mathcal{V}_F$. A natural linear ordering of V is obtained by

- *fixing B linearly ordered basis of V,*
- *fixing a linear ordering of F with least element 0_F,*
- *taking the resulting lexicographical ordering induced on V.*

$\mathcal{V}_F^{n<}$: the class of naturally ordered finite vector spaces over F.

Vector spaces, cont'd

Theorem (Thomas, 86)

- *$\mathcal{V}_F^{n<}$ is a Fraïssé order class with reduct $\mathcal{V}_F$,*
- *$\mathcal{V}_F^{n<}$ has the expansion property over $\mathcal{V}_F$.*

Theorem (Graham-Leeb-Rothschild, 72)

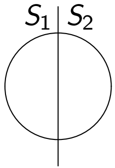
$\mathcal{V}_F^{n<}$ has the Ramsey property.

Corollary

Let $nLO(F^{<\omega})$ be the set of all linear orderings on $F^{<\omega}$ with natural restrictions on finite-dimensional subspaces. Then:
$GL(F^{<\omega}) \curvearrowright M(GL(F^{<\omega}))$ is $GL(F^{<\omega}) \curvearrowright nLO(F^{<\omega})$.

The case of $S(2)$

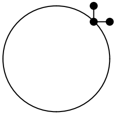
- Finite substructures of $(S(2), <)$ never have the Ramsey property: $\exists$2-coloring of the vertices with no monochromatic 3-cycle.
- Ramsey property holds if $S(2)$ is enriched differently:



- Key fact: $(S(2), S_1, S_2) \cong (\mathbb{Q}, Q_1, Q_2, <)$, Q_1, Q_2 dense subsets of $\mathbb{Q}$ (Reversing the arcs between points in different parts).
- Ramsey and expansion property hold for the corresponding finite substructures.

The case of $S(2)$, cont'd

- $\mathrm{Aut}(S(2)) \curvearrowright M(\mathrm{Aut}(S(2)))$ is $\mathrm{Aut}(S(2)) \curvearrowright \overline{\mathrm{Aut}(S(2)) \cdot (S_1, S_2)}$.
- $\overline{\mathrm{Aut}(S(2)) \cdot (S_1, S_2)} \cong (\mathbb{S}^1$ with rational and corational points doubled).
- Thus, $\mathrm{Aut}(S(2)) \curvearrowright M(\mathrm{Aut}(S(2)))$ is $\mathrm{Aut}(S(2)) \curvearrowright (\mathbb{S}^1$ with rational and corational points doubled):



Part VI

Perspectives

Down to earth

- Exploit further the equivalences between combinatorics and topological dynamics.
  - From combinatorics to topological dynamics (all examples so far)...
    ...The problem here is that proving the Ramsey property is usually difficult.
  - But also the other way around! Use dynamics to prove new Ramsey-type results!
    ...The problem here is that nobody really knows how to attack extreme amenability for closed subgroups of S_∞.
- Even when there is an extremely amenable group (not necessarily closed subgroup of S_∞) around, going back to combinatorics is not easy. Typical example: Gromov-Milman theorem:
  Is there a Ramsey theorem for finite ordered affinely independent Euclidean metric spaces, distances in $\mathbb{Q}$?
- Is metrizability of $M(\mathrm{Aut}(\mathbb{F}))$ equivalent to existence of a finite relational expansion $\mathbb{F}^*$ with Ramsey and expansion property?

General

- Is there a unified approach to prove Ramsey property for classes of finite structures?
- How far can computations of universal minimal flows presented here go? Can it help to capture the case of concrete homeomorphism groups like $Homeo(\mathbb{S}^2)$ or $Homeo([0,1]^{\mathbb{N}})$?
- Recent developments of the theory to attack those last questions:
  - Projective version (Irwin-Solecki).
  - Dual version (Solecki).
  - Relational Polish metric structures (Ben Yaacov, Melleray, Tsankov).
- Systematize the transfer between finite combinatorics on Fraïssé classes (resp. any of the versions above) and groups (recent advances by Kechris-Rosendal, Tsankov).

Thank you very much for your attention!