Uniformity of the van der Waerden ideal

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AP-sets and van der Waerden theorem

Definition.

A set $A \subseteq \mathbb{N}$ is called an AP-set if it contains arbitrary long arithmetic progressions.

Van der Waerden Theorem.

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Sets which are not AP-sets form a proper ideal on $\mathbb{N}$
— van der Waerden ideal denoted by $\mathcal{W}$

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$$\mathcal{W}_n = \{A \subseteq \mathbb{N} : A \text{ contains no a. p. of length } n\}$$
- not a P-ideal — consider for example the sets
$$A_k = \{2^n + k : n \in \omega\} \text{ for } k \in \omega$$

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Szemerédi Theorem.

$$\mathcal{W} \subseteq \mathcal{Z} \quad \text{where} \quad \mathcal{Z} = \{A \subseteq \mathbb{N} : \limsup_{n\to\infty} \frac{|A \cap n|}{n} = 0\}$$

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Erdős Conjecture.

$$\mathcal{W} \subseteq \mathcal{I}_{1/n} \quad \text{where } \mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

Cardinal invariants of ideals on ω

Definition (Hernández-Hernández, Hrušák).

Let $\mathcal{I}$ be a tall ideal on ω containing the ideal of finite sets. Define the following cardinals associated with $\mathcal{I}$:

$$\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(A \not\subseteq^* I)\}$$

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\mathrm{non}^*(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq [\omega]^{\aleph_0} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(|A \cap I| < \aleph_0)\}
\end{aligned}$$

Cardinal invariants of ideals on ω

The inequalities holding among these cardinals are summarized in the following diagram:

$$\begin{tikzcd}
 & & \mathrm{cov}^*(\mathcal{I}) \arrow[dr] & & \\
\aleph_0 \arrow[r] & \mathrm{add}^*(\mathcal{I}) \arrow[ur] \arrow[dr] & & \mathrm{cof}^*(\mathcal{I}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathrm{non}^*(\mathcal{I}) \arrow[ur] & &
\end{tikzcd}$$

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2. $\omega = \bigcup_{n\in\omega} I_n$ where $I_n = [2^n; 2^{n+1})$
3. For every $A \in \mathcal{A} \subseteq [\omega]^{\aleph_0}$ define $f_A : \omega \to \omega$

$$f_A = \begin{cases} \min(I_n \cap A) & \text{if } I_n \cap A \neq \emptyset \\ \text{undefined} & \text{otherwise} \end{cases}$$

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5. $I = \{g(n) : n \in \omega\} \in \mathcal{W}$ and $|I \cap A| = \aleph_0$ for every $A \in \mathcal{A}$

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4. Show that for every $I \in \mathcal{I}$ there exists $R \in \mathcal{R}$, $k \in \mathbb{N}$ with
$$|A_{R,k} \cap I| < \aleph_0$$

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$$\text{non}^*(\mathcal{Z}) \leq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$$

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Corollary 3. $\mathrm{non}^*(\mathcal{W}) \leq \max\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$

Questions about upper bounds for $\mathrm{non}^*(\mathcal{W})$

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NO. In the dual Hechler model $\mathfrak{d} = \aleph_1$ and $\mathrm{non}^*(\mathcal{W}) = \aleph_2$.

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VERY LIKELY YES.

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Question B. Does $\mathrm{non}^*(\mathcal{W}) \leq \mathrm{non}(\mathcal{N})$ hold in ZFC?

VERY LIKELY YES. Because $\mathrm{non}^*(\mathcal{I}_{1/n}) \leq \mathrm{non}(\mathcal{N})$ (H.-H., Hr.) and $\mathrm{non}^*(\mathcal{W}) \leq \mathrm{non}^*(\mathcal{I}_{1/n})$ if Erdős Conjecture is true.

Questions about lower bounds for $\mathrm{non}^*(\mathcal{W})$

Theorem (Hernández-Hernández, Hrušák).

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Theorem (Hernández-Hernández, Hrušák).

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Question C. Does $\mathrm{non}^*(\mathcal{W}) \geq \min\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ hold in ZFC?

What about other small cardinals — $\mathfrak{b}$, $\mathfrak{h}$, $\mathfrak{p}$ etc.?

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Observation 4. $\mathrm{add}^*(\mathcal{W}) = \aleph_0$

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Proposition 5. $\mathrm{cof}^*(\mathcal{W}) = 2^{\aleph_0}$

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Sketch of the proof:

1. Show that there exists a perfect set $P \subseteq {}^\omega\omega$ such that every $f \in P$ satisfies $f(n+1) > 2f(n)$ for every $n \in \omega$ and whenever $f_0, f_1, \ldots f_k \in P$ are distinct, there exist infinitely many $n \in \omega$ such that $\{f_0(n), f_1(n), \ldots f_k(n)\}$ is a set of $k+1$ successive integers.

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2. $A_f = \{f(n) : n \in \omega\} \in \mathcal{W}$ for every $f \in P$

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2. $A_f = \{f(n) : n \in \omega\} \in \mathcal{W}$ for every $f \in P$
3. $\{f \in P : A_f \subseteq^* B\}$ is finite for every $B \in \mathcal{W}$

Covering number of $\mathcal{W}$

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Conjectures.

1. $\mathrm{cov}^*(\mathcal{W}) \leq \mathrm{non}(\mathcal{M})$
2. $\mathrm{cov}^*(\mathcal{W}) \geq \mathfrak{s}$
3. $\mathrm{cov}^*(\mathcal{W}) \leq \max\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$

and many more...

References

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