Almost disjoint families and C*-algebras

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Outline

Definition

A C*-algebra $\mathcal{A}$ is a structure $(\mathcal{A}, +, ., ., *, \|\|)$ such that

1. $(\mathcal{A}, +, ., ., \|\|)$ is a Banach algebra over $\mathbb{C}$,
2. $(a+b)^* = a^* + b^*$, $(\alpha a)^* = \bar{\alpha} a^*$, $(ab)^* = b^* a^*$,
3. $\|aa^*\| = \|a\|^2$ (the C^*-identity),

for every $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

Examples

- $M_n(\mathbb{C})$,
- $\mathcal{B}(\mathcal{H})$ - The C*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$,
- $\mathcal{K}(\mathcal{H})$ - The ideal of all compact operators on $\mathcal{H}$,

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Examples

- For a locally compact Hausdorff space X, the space $C_0(X)$ with

$$f.g(x) = f(x)g(x),$$

$$f^*(x) := \overline{f(x)},$$

$$\|f\| = \sup\{f(x) : x \in X\},$$

is a commutative C*-algebra.

Theorem (Gelfand)

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Almost disjoint families

Recall that an almost disjoint family $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\} \subseteq \mathcal{P}(\mathbb{N})$ is called Luzin if for every $\alpha < \omega_1$ and $n \in \mathbb{N}$

$$\{\beta < \alpha : D_\alpha \cap D_\beta \subseteq n\}$$

is finite.

Facts

- There are Luzin families in ZFC.
- There are no separations of uncountable subfamilies i.e., given two disjoint uncountable $\mathcal{D}', \mathcal{D}'' \subseteq \mathcal{D}$ there is no $X \subseteq \mathbb{N}$ such that $A \subseteq^* X$ and $B \cap X =^* \emptyset$ for all $A \in \mathcal{D}'$ and $B \in \mathcal{D}''$.

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Almost disjoint families in C^*-algebras

- $p \in \mathcal{A}$ is a projection if $p^2 = p^* = p$.

Fix an orthonormal basis $\{e_x : x \in \mathbb{N}\}$ for ℓ_2. For every $A \subseteq \mathbb{N}$ let P_A denote the projection on the closed subspace spanned by $\{e_n : n \in A\}$.

$$P_A P_B \in \mathcal{K}(\ell_2) \iff A \cap B \in Fin.$$

Definition (Wofsey)

For a Hilbert space $\mathcal{H}$, a family $\mathcal{P}$ of noncompact projections of $\mathcal{B}(\mathcal{H})$ is called almost orthogonal if the product of any two distinct elements is compact.

- For an almost disjoint family $\mathcal{D} = \{A_\xi : \xi < \kappa\}$, the corresponding family $\{P_{A_\xi} : \xi < \kappa\}$ is an almost orthogonal family of projections.

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- SOME APPLICATIONS

lifting projections of the Calkin algebra

$\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H) :=$ The Calkin algebra.

Fact

Every countable commuting family of projections of the Calkin algebra can be simultaneously lifted to a family of commuting projections in $\mathcal{B}(H)$.

Theorem (Anderson, 1979)

Under CH there is an uncountable family $\mathcal{P}$ of commuting projections in the Calkin algebra such that no uncountable $\mathcal{P}_1 \subseteq \mathcal{P}$ can be simultaneously lifted to a family of commuting projections in $\mathcal{B}(H)$.

Theorem (Farah 2006, Bice-Koszmider 2016)

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Sketch of the proof

Fix a dense set of operators $\{K_n : n \in \mathbb{N}\} \subseteq \mathcal{K}(H)$ and $0 < \epsilon < 1/2$.

Recursively construct a (Luzin-like) family of projections $\{P_\xi : \xi < \omega_1\}$ in $\mathcal{B}(H)$:

1. $P_\xi P_\eta \in \mathcal{K}(H)$ for all $\xi \neq \eta$,
2. for every $\alpha < \omega_1$ and for all n, the set of all $\beta < \alpha$ such that

$$\|(P_\alpha - K_n)(P_\beta - K_n) - (P_\beta - K_n)(P_\alpha - K_n)\| < \epsilon$$

is finite.

$\Rightarrow$ No uncountable subset of $\{P_\xi : \xi < \omega_1\}$ can be perturbed by compact operators to commuting ones.

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Definition

A locally compact space K is called scattered if every nonempty subset of K has an isolated point. Equivalently every continuous image of K has an isolated point.

Definition (Cantor-Bendixon Derivatives)

- $K^{(1)} = K'$ be the set of all non-isolated points of K,
- $K^{(\alpha+1)} = K^{(\alpha)\prime}$,
- $K^{(\gamma)} = \bigcap_{\alpha<\gamma} K^{(\alpha)}$, for limit ordinal γ.

- K is scattered iff for for an ordinal $ht(K)$ (the height of K) such that $K^{ht(K)} = \emptyset$.
- The width of K is the supremum of the cardinality of $K^{(\alpha)} \setminus K^{(\alpha+1)}$ for $\alpha < ht(K)$.

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- Scattered C^*-algebras

Definition

isolated points $\leftrightsquigarrow$ minimal projections

A projection p in $\mathcal{A}$ is called minimal if $p\mathcal{A}p = \mathbb{C}p$.

- In $\mathcal{B}(H)$ minimal projections are projections onto one dimensional subspaces.
- In $C(X)$ minimal projections correspond to the characteristic functions of isolated points of X.

Definition

A C^*-algebra $\mathcal{A}$ is called scattered if every nonzero subalgebra $\mathcal{B} \subseteq \mathcal{A}$, has a minimal projection. Equivalently every non-zero $*$-homomorphic image of $\mathcal{A}$ has a minimal projection,

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A projection p in $\mathcal{A}$ is called minimal if $p\mathcal{A}p = \mathbb{C}p$.

- In $\mathcal{B}(H)$ minimal projections are projections onto one dimensional subspaces.
- In $C(X)$ minimal projections correspond to the characteristic functions of isolated points of X.

Definition

A C^*-algebra $\mathcal{A}$ is called scattered if every nonzero subalgebra $\mathcal{B} \subseteq \mathcal{A}$, has a minimal projection. Equivalently every non-zero $*$-homomorphic image of $\mathcal{A}$ has a minimal projection,

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For a C*-algebra $\mathcal{A}$, let $\mathcal{I}^{At}(\mathcal{A})$ denote the $*$-subalgebra of $\mathcal{A}$ generated by its minimal projections.

Theorem

Suppose that $\mathcal{A}$ is a C^*-algebra.

1. $\mathcal{I}^{At}(\mathcal{A})$ *is an ideal of* $\mathcal{A}$,
2. $\mathcal{I}^{At}(\mathcal{A})$ *is isomorphic to a subalgebra of* $\mathcal{K}(\mathcal{H})$ *of compact operators on a Hilbert space* $\mathcal{H}$,
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The Cantor-Bendixson composition series

Suppose $\mathcal{A}$ is a scattered C*-algebra. We define the Cantor-Benndixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$ of ideals of $\mathcal{A}$ by induction:

- $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_{ht(\mathcal{A})} = \mathcal{A}$,
- $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha = \mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$, for $\alpha < ht(\mathcal{A})$,
- If γ is a limit ordinal let $\mathcal{I}_\gamma = \overline{\bigcup_{\alpha<\gamma} \mathcal{I}_\alpha}$.

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Examples

- $\mathcal{K}(H)$,
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- Ψ-type C^*-algebras

Ψ-spaces (Mrówka-Isbell spaces)

Let $\mathcal{D} = \{A_\xi : \xi < \kappa\}$ be an almost disjoint family of subsets of $\mathbb{N}$.
The $\Psi(\mathcal{D})$ is the space $\mathbb{N} \cup \mathcal{D}$, where all elements of $\mathbb{N}$ are isolated and the basic neighborhoods of $A_\xi \in \mathcal{D}$ are of the form $\{A_\xi\} \cup A_\xi \setminus F$ for some finite set $F \subseteq \mathbb{N}$.

$\Psi(\mathcal{D})$ is a separable, scattered space of height two.

Faithfully represent $C_0(\Psi(\mathcal{D}))$ in $\mathcal{B}(\ell_2)$, by $\pi : C_0(\Psi(\mathcal{D})) \to \mathcal{B}(\ell_2)$

$$\pi(\chi_{\{n\}}) = Proj\ span\{e_n\},$$
$$\pi(\chi_{A_\xi}) = Proj\ \overline{span\{e_n : n \in A_\xi\}}.$$

Let $P_\xi = \pi(\chi_{A_\xi})$.

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- $\mathcal{A}_{\mathcal{D}} = \pi[C(\Psi(\mathcal{D}))]$ is the commutative C*-algebra generated by $\{P_\xi : \xi < \kappa\}$ and $c_0 \subseteq \mathcal{K}(\ell_2)$.
- For $\xi \neq \eta$, we have $P_\xi P_\eta \in \mathcal{K}(\ell_2)$, since $A_\xi \cap A_\eta$ is finite.
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$$\mathcal{I}_1 = c_0, \qquad \mathcal{I}_2/\mathcal{I}_1 \cong c_0(\kappa), \qquad \mathcal{I}_2 = C_0(\Psi(\mathcal{D})).$$

Hence

$$0 \to c_0 \xhookrightarrow{\iota} C_0(\Psi(\mathcal{D})) \to c_0(\kappa) \to 0,$$

We would like to obtain a non-commutative version of this phenomena, i,e, a C*-algebra $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ which contains $\mathcal{K}(\ell_2)$ as an (essential) ideal and satisfies

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In order to build such $\mathcal{A}$,

- Start with $\mathcal{A}_{\mathcal{D}}$,
- add $\mathcal{K}(\ell_2)$,
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Definition

We say $\mathcal{T} = \{T_{\xi,\eta} : \xi, \eta < \kappa\} \subseteq \mathcal{B}(\ell_2(\kappa))$ is a system of matrix units if and only if for every $\alpha, \beta, \xi, \eta < \kappa$,

1. $T^*_{\eta,\xi} = T_{\xi,\eta}$,
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Theorem (Mrówka)

There is a maximal almost disjoint family $\mathcal{D}$ of size $\mathfrak{c}$, such that $|\beta(\Psi(\mathcal{D})) \setminus \Psi(\mathcal{D})| = 1.$

Compactifications $\leftrightsquigarrow$ Unitizations
One-point Compactification $\leftrightsquigarrow$ The (minimal) unitization
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Theorem (G., Koszmider, 2016)

There is a system of almost matrix units $\mathcal{S}$ of size $\mathfrak{c}$ such that $\mathcal{A}(\mathcal{S})$ has the property that the multiplier algebra $\mathcal{M}(\mathcal{A}(\mathcal{S}))$ of $\mathcal{A}(\mathcal{S})$ is isomorphic to the (minimal) unitization of $\mathcal{A}(\mathcal{S})$, i.e., $\mathcal{M}(\mathcal{A}(\mathcal{S}))/\mathcal{A}(\mathcal{S}) \cong \mathbb{C}$.

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There is a system of almost matrix units $\mathcal{S}$ of size $\mathfrak{c}$ such that $\mathcal{A}(\mathcal{S})$ has the property that the multiplier algebra $\mathcal{M}(\mathcal{A}(\mathcal{S}))$ of $\mathcal{A}(\mathcal{S})$ is isomorphic to the (minimal) unitization of $\mathcal{A}(\mathcal{S})$, i.e., $\mathcal{M}(\mathcal{A}(\mathcal{S}))/\mathcal{A}(\mathcal{S}) \cong \mathbb{C}$.

Theorem (Mrówka)

There is a maximal almost disjoint family $\mathcal{D}$ of size $\mathfrak{c}$, such that $|\beta(\Psi(\mathcal{D})) \setminus \Psi(\mathcal{D})| = 1$.

Compactifications $\leftrightsquigarrow$ Unitizations
One-point Compactification $\leftrightsquigarrow$ The (minimal) unitization
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Definition

A C*-algebra $\mathcal{A}$ is called stable, if $\mathcal{A} \otimes \mathcal{K}(\ell_2) \cong \mathcal{A}$.

- For any infinite-dimensional Hilbert space $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ is stable, since $\mathcal{K}(\mathcal{H}) \otimes \mathcal{K}(\ell_2) \cong \mathcal{K}(\mathcal{H} \otimes \ell_2) \cong \mathcal{K}(\mathcal{H})$.
- The Mrówka C*-algebra $\mathcal{A}(\mathcal{S})$ is not stable, since

$$\mathcal{B}(\ell_2) \hookrightarrow \mathcal{M}(\mathcal{A}(\mathcal{S})) \otimes \mathcal{B}(\ell_2) \subseteq \mathcal{M}(\mathcal{A}(\mathcal{S}) \otimes \mathcal{K}(\ell_2)).$$

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Extensions of C^*-algebras

Fact

It is well-known (Brown- Douglas-Fillmore) that for an extension

$$0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} \mathcal{A} \xrightarrow{\pi} \mathcal{B} \to 0$$

for separable $\mathcal{A}$ and $\mathcal{B}$, the C*-algebra $\mathcal{A}$ is stable if and only if $\mathcal{B}$ is stable.

Not true for non-separable C*-algebras, since

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Thin-tall C^*-algebras

A locally compact scattered space K is called thin-tall if $ht(K) = \omega_1$, $wd(K) = \omega$.

- In 1978 Juhász abd Weiss showed the existence of a compact thin-tall space.
- Simon and Weese were first to construct two nonisomorphic compact thin-tall spaces.
- Dow and Simon showed that in ZFC, there are 2^{ω_1} (as many as possible) pairwise non-isomorphic compact thin-tall spaces.

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Definition

A C^*-algebra $\mathcal{A}$ is called fully noncommutative thin-tall if there is a sequence of ideals $(\mathcal{I}_\alpha)_{\alpha \leq \omega_1}$ of $\mathcal{A}$ is such that

1. $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_{\omega_1} = \mathcal{A}$, $\mathcal{I}_\alpha \subseteq \mathcal{I}_{\alpha'}$ for $\alpha \leq \alpha' \leq \omega_1$,
2. $\mathcal{I}_\lambda = \overline{\bigcup_{\alpha<\lambda} \mathcal{I}_\alpha}$ for all limit ordinals $\lambda \leq \omega_1$,

   For every $\alpha < \omega_1$
3. $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is an essential ideal of $\mathcal{A}/\mathcal{I}_\alpha$,
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Theorem (G, Koszmider, 2017)

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The idea

Construct a Luzin-like sequence $(\mathcal{A}_\alpha)_{\alpha<\omega_1}$ of C^*-subalgebras of $\mathcal{B}(\ell_2)$ such that

- $\mathcal{A}_\alpha \cong \mathcal{K}(\ell_2)$
- they are pairwise almost orthogonal, i.e., $AA' =^{\mathcal{K}} 0$ for all $A \in \mathcal{A}_\alpha$, $A' \in \mathcal{A}_{\alpha'}$ for any $\alpha < \alpha' < \omega_1$,
- Given any two uncountable $X, Y \subseteq \omega_1$ and any choice of $A_\alpha \in \mathcal{A}_\alpha$ for $\alpha \in X$ and $B_\beta \in \mathcal{A}_\beta$ for $\beta \in Y$ there is no projection $P \in \mathcal{B}(\ell_2)$ satisfying

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Thank you