Cichoń’s Maximum

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Outline

Background

Generics over subuniverses

Linear witnesses and cone witnesses

Boolean ultrapowers

Proof ideas
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- Boolean ultrapowers

#### Proof ideas
Cichoń’s Diagram

\[ \mathcal{M} = \text{the ideal of meager subsets of } \mathbb{R}. \]

\[ \mathcal{N} = \text{the ideal of Lebesgue null sets of } \mathbb{R}. \]

\begin{align*}
\text{cov}(\mathcal{N}) & \to \text{non}(\mathcal{M}) \to \text{cof}(\mathcal{M}) \to \text{cof}(\mathcal{N}) \to 2^{\aleph_0} \\
\aleph_1 & \to \text{add}(\mathcal{N}) \to \text{add}(\mathcal{M}) \to \text{cov}(\mathcal{M}) \to \text{non}(\mathcal{N}) \end{align*}

Are these cardinals different?
Examples

- CH $\iff$ all these cardinals are equal.
- MA $\land \neg$CH $\Rightarrow$ 2 values: $\aleph_1 < \text{add}(\mathcal{N}) = 2^{\aleph_0}$.
- Many other consistency results for 2 values. e.g.

\[
\begin{array}{cccccc}
\square & \rightarrow & \blacksquare & \rightarrow & \cdot & \rightarrow & \blacksquare & \rightarrow & \blacksquare \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\blacksquare & \rightarrow & \blacksquare & & \blacksquare & & \blacksquare & & \blacksquare \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\square & \rightarrow & \square & \rightarrow & \cdot & \rightarrow & \square & \rightarrow & \square \\
\end{array}
\]

- Many consistency results for more than 2 values.
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\begin{equation*}
\begin{tikzcd}
\text{cov}(\mathcal{N}) \arrow{r} & \text{non}(\mathcal{M}) \arrow{r} & \text{cof}(\mathcal{M}) \arrow{r} & \text{cof}(\mathcal{N}) \arrow{r} & 2^{\aleph_0} \\
& & & \text{b} \arrow{ur} & \text{d} \arrow{ur} & \\
\aleph_1 \arrow{u} \arrow{r} & \text{add}(\mathcal{N}) \arrow{u} \arrow{r} & \text{add}(\mathcal{M}) \arrow{u} \arrow{r} & \text{cov}(\mathcal{M}) \arrow{u} \arrow{r} & \text{non}(\mathcal{N}) \arrow{u}
\end{tikzcd}
\end{equation*}

In ZFC:

\begin{align*}
\text{add}(\mathcal{M}) &= \min(\text{b}, \text{cov}(\mathcal{M})) \\
\text{cof}(\mathcal{M}) &= \max(\text{non}(\mathcal{M}), \text{d})
\end{align*}
The left side

General strategy: E.g., to get $\text{cov}(\mathcal{N}) \geq \lambda_2$, iterate (with finite support) for a long time, and make sure to take care of all “small” families $F$ of measure zero sets by adding a random real over $F$. (“small” means: $< \lambda_2$.)

Hopefully, will not make $\text{cov}(\mathcal{N}) > \lambda_2$.

For simplicity, we will today only consider $\text{cov}(\mathcal{N})$ and $\mathfrak{b}$ on the left side, $\mathfrak{d}$ and $\text{non}(\mathcal{N})$ on the right side.
Main theorem

(G-Kellner-Shelah 2017, arXiv:1708.03691)

Starting from a universe with 4 strongly compact cardinals, we construct a universe in which 10 values $\aleph_1 = \lambda_0 < \cdots < \lambda_9 = 2^{\aleph_0}$ appear in Cichon’s diagram:

\[
\begin{array}{cccccc}
\lambda_2 & \rightarrow & \lambda_4 & \rightarrow & \cdots & \rightarrow \lambda_8 & \rightarrow \lambda_9 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\lambda_3 & \rightarrow & \lambda_6 \\
\uparrow & & \uparrow \\
\lambda_0 & \rightarrow & \lambda_1 & \rightarrow & \cdots & \rightarrow \lambda_5 & \rightarrow \lambda_7
\end{array}
\]
A fragment of the main theorem

\[ \text{cov}(N) = \lambda_2 \rightarrow ?? \rightarrow \cdot \rightarrow ?? \rightarrow ?? \]

\[ b = \lambda_3 \rightarrow d = \lambda_6 \]

\[ \mathbb{N}_1 \rightarrow ?? \rightarrow \cdot \rightarrow ?? \rightarrow \text{non}(N) = \lambda_7 \]
How to make $b$ large, say: $b \geq \lambda$?

- Iterate a long time.
  In each step add a real dominating some set of size $< \lambda$.
  Use bookkeeping.
  So every small set will be dominated.

How to make $b$ small, say: $b \leq \lambda$.

- Iterate $\lambda$ steps (or at least with cofinality $\lambda$).
  In each step add an unbounded real.
  The generic reals will be an unbounded set.
How to ensure $b \geq \lambda_3$

A “standard” iteration is a FS (finite support) iteration
$\bar{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ of ccc forcing notions together with a
bookkeeping device $\bar{w} = (w_\alpha : \alpha < \delta)$, where:

- $\bar{w}$ is cofinal in $[\delta]^{<\lambda_3}$, and $\forall \alpha < \delta: w_\alpha \subseteq \alpha$
- $Q_\alpha$ adds a new generic $c_\alpha$ over $V^{\bar{P}|w_\alpha}$.
  (A dominating real if we want to get $b \geq \lambda_3$)
- $V^{\bar{P}|w_\alpha}$ is the model computed from $(c_\beta : \beta \in w_\alpha)$.
- To get $b \geq \lambda_3$ and $\text{cov}(\mathcal{N}) \geq \lambda_2$, let $\delta = S^2 \cup S^3$, use cofinal
  families $\{w^2_\alpha : \alpha \in S^2\} \subseteq [\delta]^{<\lambda_2}$, $\{w^3_\alpha : \alpha \in S^3\} \subseteq [\delta]^{<\lambda_3}$, add
  random reals on $S^2$ and dominating reals on $S^3$.

WARNING: This is not trivial. Usually we want $=$, not $\geq$. Some
work is needed to ensure $b \leq \lambda_3$, $\text{cov}(\mathcal{N}) \leq \lambda_2$.
Use/Develop “preservation theorems”.
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Witnesses

- A witness for \( d \leq \lambda \) is a family \((g_i : i < \lambda) \in (\omega^\omega)\) of functions \(g_i \in \omega^\omega\) such that \(\forall f \in \omega^\omega \ \exists i < \lambda : f \leq g_i\).

- A witness for \( b \geq \lambda \) is a family \((f_i : i < \lambda) \in (\omega^\omega)\) of functions \(f_i \in \omega^\omega\) such that \(\forall g \in \omega^\omega \ \exists i < \lambda : f_i \not\leq g\).

Similar definitions can be made for the other characteristics. For example, a witness for \( \text{non}(M) \leq \lambda \) is a family \((x_i : i < \lambda)\) of reals which is not meager (equivalently: for every code \(y\) of a meager Borel set \(M_y\) there is \(i < \lambda\) such that \(x_i \notin M_y\)).

In the following slides we will only deal with \(b\) and \(d\); obvious (or at least: routine) modifications will yield appropriate definitions dealing with the other characteristics.
Strong witnesses (for $b$ small and $d$ large): linear witnesses

Recall: $\mathcal{F} = (f_i : i < \lambda)$ is a witness for $b \leq \lambda$ iff $\mathcal{F}$ is unbounded:

$$\forall g \in \omega^\omega : \exists i \ f_i \not\preceq^* g$$

A linear $\lambda$-witness is a family $\mathcal{F} = (f_i : i < \lambda)$ of elements of $\omega^\omega$ such that any $g$ can only bound an initial segment of $\mathcal{F}$:

$$\forall g \in \omega^\omega : \forall^\infty i < \lambda : \ f_i \not\preceq^* g$$

($\forall^\infty i < \lambda : \cdots$ means “eventually”, i.e., $\exists i_0 \forall i \in (i_0, \lambda) : \cdots$)

$\text{LCU}_{b,d}(\lambda)$: “there is a linear witness of length $\lambda$”.

FACT: $\text{LCU}_{b,d}(\lambda) \Rightarrow b \leq \lambda, \ d \geq \lambda$.

FACT: $\text{LCU}(\lambda) \iff \text{LCU}(\text{cf}(\lambda))$.

Similarly $\text{LCU}_{\text{cov}(\mathcal{N}),\text{non}(\mathcal{N})}$.
Strong witnesses (for $b$ large and $d$ small): cone witnesses

Recall: $G = (g_j : j < \lambda)$ is a witness for $d \leq \lambda$ iff $G$ dominates:

$$\forall f \in \omega^\omega : \exists j \ f \leq^* g_j$$

Let $\lambda, \mu$ be regular uncountable. COB$_{b,d}(\lambda, \mu)$ means that there is a $(\lambda, \mu)$-cone witness: a $<\lambda$-directed partial order $(S, \leq)$ of size $\mu$ together with a sequence $(g_s : s \in S)$ of functions $g_s \in \omega^\omega$ such that

$$\forall f \in \omega^\omega \ \forall^\infty s \in S : f \leq g_s$$

As above, $\forall^\infty s \in S$ means “eventually”, i.e., $\exists s_0 \in S \ \forall s > s_0 \ldots$

FACT: COB$_{b,d}(\lambda, \mu) \Rightarrow b \geq \lambda, d \leq \mu$.

We call the set $\{s \in S \mid s \geq s_0\}$ the “cone with tip $s_0$”. If $S$ is $<\lambda$-directed, then the cones generate a $<\lambda$-closed filter.
Strong witnesses, example 1 ($b \leq \lambda$)

Example

Let $\lambda$ be regular uncountable. Let $(P_\alpha, Q_\alpha : \alpha < \lambda)$ be a finite support ccc iteration which adds (among other things) an unbounded real $c_\alpha$ at every step. Then $P_\lambda$ (the FS limit of this iteration) forces that $\vec{c} = (c_\alpha : \alpha < \lambda)$ is a linear $\lambda$-witness. (Hence, $P_\lambda$ forces that $b \leq \lambda$ and $d \geq \lambda$.)

Moreover: If $\lambda' > \lambda$, and we extend $(P_\alpha, Q_\alpha : \alpha < \lambda)$ to a longer iteration $(P_\alpha, Q_\alpha : \alpha < \lambda')$, and the forcings $Q_\alpha$ are “sufficiently nice”, then $P_{\lambda'}$ will force that $(c_\alpha : \alpha < \lambda)$ remains a linear $\lambda$-witness, and also $(c_\alpha : \alpha < \lambda')$ becomes a linear $\lambda'$-witness. (So $P_{\lambda'}$ forces $\text{LCU}_{b,\varnothing}(\lambda)$ and $\text{LCU}_{b,\varnothing}(\lambda')$, so $b \leq \lambda$ and $d \geq \lambda'$.)

Strong witnesses, example 2 \((b \geq \lambda)\)

**Example**

Let \((w_\alpha : \alpha < \delta)\) be a family of sets which is cofinal in \([\delta]^{<\lambda}\), with \(w_\alpha \subseteq \alpha\) for all \(\alpha\).

Let \((P_\alpha, Q_\alpha : \alpha < \delta)\) be a “standard” finite support ccc iteration designed to make \(b \geq \lambda\), based on \((w_\alpha : \alpha \in S) \subseteq [\delta]^{<\lambda} , S \subseteq \delta\) (each \(Q_\alpha\) introduces a dominating \(c_\alpha\) over \(V^{P}\upharpoonright w_\alpha\).

Then in \(V^{P_\delta}\), the sequence \((c_\alpha : \alpha \in S)\) is a \((\lambda, |S|)\)-cone witness. So we have \(\text{COB}_{b,0}(\lambda, |S|)\), so \(b \geq \lambda\), and \(0 \leq |S|\).

(We order \(S\) by \(\alpha \sqsubseteq \beta \iff w_\alpha \subseteq w_\beta\). This partial order is clearly \(<\lambda\)-directed. Every \(P_\delta\)-name of a real uses only few coordinates, hence will be in “almost all” \(V^P\upharpoonright W_\alpha\), therefore dominated by almost all \(c_\alpha\).)
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Boolean ultrapowers (bups)

Let $B$ be a $\kappa$-distributive Boolean algebra with the $\kappa^+$-cc. A $B$-bup-name is an pair $(A, f)$, where $A$ is a maximal antichain in $B$ and $f : A \to V$.

Essentially: A $B$-bup-name is the same as a name of an element of $V$, using $B$ as a forcing notion. If $\tau$ and $\sigma$ are $B$-names we write $[\tau = \sigma]$ for the Boolean value of the statement $\tau = \sigma$.

Let $U$ be a $<\kappa$-complete ultrafilter on $B$. (So $U$ meets all maximal antichains of $B$ of size $< \kappa$, but in general not all those of size $\kappa$.) Then $U$ defines an equivalence relation $\tau \sim_U \sigma \iff [\tau = \sigma] \in U$.

The Boolean ultrapower $M = V^B/U$ is the set of all $\sim_U$-equivalence classes (after the Mostowski collapse). There is a natural embedding $j : V \to M$ using standard names.
Boolean ultrapowers, examples

Example
Let $B$ be a complete Boolean algebra, and let $U \subseteq B$ be a $V$-generic ultrafilter. Then every element of $M$ is of the form $j(x)$, for some $x \in V$. So $M = V$, $j = id$.

Example
Let $B = \mathcal{P}(\kappa)$ be the powerset of $\kappa$. Then every antichain can be refined to the antichain $\{\alpha\} : \alpha \in \kappa$, so every $B$-bup-name is equivalent to a function $f : \kappa \to V$. In this case $M$ is the “traditional” ultrapower $V^\kappa/U$. 
Boolean ultrapower embeddings

Assume GCH. Assume that $\kappa$ is strongly compact. Then for every regular $\theta > \kappa$ there is an elementary embedding $j : V \rightarrow M$ with the following properties:

- $\kappa = cp(j)$
- $\theta \leq j(\kappa) \leq \theta^+$.
- (Every $x \in M$ is described by some $(A, f)$ of size $\kappa$)
- If $(S, <)$ is $<\kappa^+$-directed in $V$, then $j'' S$ is cofinal in $j(S)$.
- If $\lambda \neq \kappa$ is regular, then $cf(j(\lambda)) = \lambda$.
- If $\vec{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ is a FS ccc iteration, then $j(\vec{P})$ is a FS ccc iteration of length $j(\delta)$ not only in $M$, but also in $V$.

Note: $M$ is $\leq \kappa$-closed. Contains all reals, even all names for reals.

REMARK: Moti Gitik suggested an extender ultrapower with a smaller large cardinal.
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Assume $P$ forces not only $\text{cov}(\mathcal{N}) = \lambda_2$, $b = \lambda_3$, $d = \lambda_6$, but moreover:

\[
\begin{align*}
\text{LCU}_{b,d}(\lambda_3), & \quad \text{LCU}_{b,d}(\lambda_6), \quad \forall \lambda \in (\lambda_2, \lambda_3) : \text{LCU}_{\text{cov}(\mathcal{N}), \text{non}(\mathcal{N})}(\lambda) \\
& \quad b \leq \lambda_3 \quad d \geq \lambda_6 \\
\text{COB}_{b,d}(\lambda_3, \lambda_6), & \quad \text{COB}_{\text{cov}(\mathcal{N}), \text{non}(\mathcal{N})}(\lambda_2, \lambda_6) \\
& \quad b \geq \lambda_3, \quad d \leq \lambda_6 \quad \text{cov}(\mathcal{N}) \geq \lambda_2, \quad \text{non}(\mathcal{N}) \leq \lambda_6
\end{align*}
\]

Assume that $\kappa$ is strongly compact, $\lambda_2 < \kappa < \lambda_3$. Let $j : V \to M$ be elementary with $\text{cp}(j) = \kappa$ and $\text{cf}(j(\kappa)) = \lambda_7$. Then $j(P) \models \text{non}(\mathcal{N}) = \lambda_7$. (And the other cardinals stay.)
Proof sketch

\[ \lambda_2 < \kappa < \lambda_3, \ cf(j(\kappa)) = \lambda_7. \]

- **b stays \leq \lambda_3:**
  \[ P \models \text{LCU}_{b, \emptyset}(\lambda_3), \ \text{so} \ j(P) \text{ forces } \text{LCU}_{b, \emptyset}(j(\lambda_3)). \]
  \[ b \leq \lambda_3 \quad b \leq j(\lambda_3) \]
  But \ \text{LCU}_{b, \emptyset}(\mu) \iff \text{LCU}_{b, \emptyset}(cf(\mu)), \ \text{so} \ j(P) \models b \leq \lambda_3. \]

- **b stays \geq \lambda_3:**
  \[ P \models \text{COB}_{b, \emptyset}(\lambda_3, \lambda_6), \ \text{so} \ j(P) \models \text{COB}_{b, \emptyset}(\lambda_3, \lambda_6). \]
  \[ b \geq \lambda_3 \quad b \geq \lambda_3 \]
  (Use \( j'' S \) as a witness! Isomorphic to \( S \), hence same size \( \lambda_6 \).)

- **non(\( \mathcal{N} \)) becomes large:**
  \[ P \models \forall \lambda \in (\lambda_2, \lambda_3) : \ldots, \text{in particular} \]
  \[ P \models \text{LCU}_{\text{cov}(\mathcal{N}), \text{non}(\mathcal{N})}(\kappa), \ \text{so} \ j(P) \models \text{LCU}_{\text{cov}(\mathcal{N}), \text{non}(\mathcal{N})}(j(\kappa)). \]
  \[ \text{non}(\mathcal{N}) \geq cf(\kappa) \quad \text{non}(\mathcal{N}) \geq cf(j(\kappa)) \]