Rigid Collapse

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Winter School in Abstract Analysis
January 30, 2018

Motivation

A mathematical structure is *rigid* if it has no automorphisms besides the identity map.

Theorem (Woodin, Larson, after Foreman-Magidor-Shelah)

MM implies that the boolean algebra $\mathcal{P}(\omega_1)/\operatorname{NS}$ is rigid.

Idea: MM implies NS_{ω_1} is *saturated*, and thus forcing with $\mathcal{P}(\omega_1)/\operatorname{NS}$ generates an elementary embedding $j : V \to M \subseteq V[G]$, where $M^\omega \cap V[G] \subseteq M$.

The forcing codes information into the manipulation of sufficiently absolute properties, which correlate to the details of the embedding, so that only one embedding can exist.

Prior results

Theorem (Cody-E.)

If GCH holds and there is a saturated ideal on $\kappa = \mu^+$, where μ is regular, then there is a μ-closed, κ-c.c. forcing extension satisfying $2^\mu = \kappa^+$ and an analogue of MA, where the generated ideal is rigid and saturated.

Theorem (Cody-E.)

If κ is almost-huge and $\mu < \kappa$ is regular and uncountable, then there is a μ-distributive forcing extension satisfying GCH + $\kappa = \mu^+$ + "There is a rigid saturated ideal on κ."

Using the same technique as above (coding into stationarity on μ), we showed:

Theorem (Cody-E.)

The existence of a rigid precipitous ideal on ω_2 is equiconsistent with a measurable cardinal.

Main result

Theorem (E.)

It is consistent relative to a huge cardinal to have GCH + "Every successor cardinal carries a rigid saturated ideal."

More generally:

Theorem (E.)

Suppose κ is a Mahlo cardinal and $\mu < \kappa$ is regular. Then there is a μ-directed-closed, κ-c.c. partial order $\mathrm{RigCol}(\mu, \kappa) \subseteq V_\kappa$ forcing $\kappa = \mu^+$, and whenever $G \subseteq \mathrm{RigCol}(\mu, \kappa)$ is generic over V, then in $V[G]$, G is the unique filter which is $\mathrm{RigCol}(\mu, \kappa)$-generic over V.

Splitting– a Σ_1 property!

Suppose $V \subseteq W$ are models of set theory.

$$\mathrm{Spl}(\mu, \kappa, V): \; (\exists A \in [\kappa]^\kappa)\,(\forall x \in [\kappa]^\mu \cap V)(\forall \alpha < \mu)(\exists y, z \in [x]^{<\mu} \cap V) \\ \min\{\mathrm{ot}(y), \mathrm{ot}(z)\} \geq \alpha, y \subseteq A, \text{ and } z \cap A = \emptyset.$$

Lemma

Suppose $\mu < \kappa$ are regular. Then $\mathrm{Col}(\mu, {<}\kappa)$ forces $\mathrm{Spl}(\mu, \kappa)$.

Lemma

Suppose $\nu < \mu < \kappa$ are regular and $\alpha^{<\nu} < \kappa$ for all $\alpha < \kappa$. Then:

1. $\Vdash_{\mathrm{Col}(\nu, <\kappa)} \neg\, \mathrm{Spl}(\mu, \kappa)$.
2. $\Vdash_{\mathrm{Col}(\mu, <\kappa)} \neg\, \mathrm{Spl}(\nu, \kappa)$.

Skipping coordinates

Lemma

Suppose κ is Mahlo. Let $X \subseteq \kappa$ be a set of regular cardinals such that for some regular $\mu < \kappa$, $\mu^+ \notin X$. Then the partial order

$$\mathbb{P} = \prod_{\alpha \in X}^{E} \mathrm{Col}(\alpha, {<}\kappa)$$

is κ-c.c. and forces $\neg\, \mathrm{Spl}(\mu^+, \kappa)$.

Proof sketch:

$$\mathbb{P} \cong \prod_{\alpha \in [0,\mu] \cap X}^{E} \mathrm{Col}(\alpha, {<}\kappa) \times \prod_{\alpha \in [\mu^{++},\kappa) \cap X}^{E} \mathrm{Col}(\alpha, {<}\kappa) := \mathbb{P}_0 \times \mathbb{P}_1$$

Skipping coordinates

$\mathbb{P}_1$ is μ^{++}-closed and κ-c.c. Let $G_1 \subseteq \mathbb{P}_1$ be generic and work in $V[G_1]$.

Suppose $q \Vdash_{\mathbb{P}_0}^{V[G_1]} \dot{A} \in [\kappa]^\kappa$. When possible, let $p_\alpha \leq q$ be such that $p_\alpha \Vdash \alpha \in \dot{A}$. Let $\langle \alpha_i : i < \kappa \rangle$ enumerate the set of α for which p_α is defined.

Since the set of ordinals below κ which were regular in V remains stationary in $V[G_1]$, we can find a stationary $S \subseteq \kappa$ such that $\{p_{\alpha_i} : i \in S\}$ forms a Δ-system with root $r \leq q$.

For every $z \in [S]^\mu$ and every $s \leq r$, $s \nVdash \{\alpha_i : i \in z\} \cap \dot{A} = \emptyset$, since $|s| < \mu$.

This shows that $\mathbb{P}_0$ forces $\neg\,\mathrm{Spl}(\mu^+, \kappa, V[G_1])$ over $V[G_1]$. Since $([\kappa]^{\mu^+})^V = ([\kappa]^{\mu^+})^{V[G_1]}$, $\mathbb{P}$ forces $\neg\,\mathrm{Spl}(\mu^+, \kappa, V)$. $\square$

Construction of RigCol(μ, κ)

Suppose κ is Mahlo and $\mu < \kappa$ is regular. Let

$$\mathbb{P} = \prod^{E}_{\alpha \in [\mu,\kappa) \cap \mathrm{Reg}} \mathrm{Col}(\alpha, {<}\kappa).$$

$\mathbb{P}$ can be viewed as a set of partial functions $p : \kappa^3 \to \kappa$. A generic for any suborder of $\mathbb{P}$ is determined by a subset of κ via the Gödel ordering on κ^4.

$$\begin{aligned} A_0 = \{\alpha \in [\mu, \kappa) : \ & \alpha = \mu, \text{ or } \alpha \text{ is inaccessible, or } \alpha = \beta^{+n} \\ & \text{for some singular cardinal } \beta \text{ of cofinality } > \mu \\ & \text{and some finite } n > 0\} \times \kappa \times \kappa. \end{aligned}$$

For $n > 0$, let A_n be the set

$$\{\alpha \in [\mu, \kappa) : \alpha = \beta^{+n+1}, \text{ for some singular cardinal } \beta \text{ of cofinality } \mu\} \times \kappa \times \kappa.$$

For $n < \omega$, let $\mathbb{Q}_n = \mathbb{P} \restriction A_n$. Note that $\mathbb{P} \restriction \bigcup_{n \in \omega} A_n \cong \prod_{n<\omega} \mathbb{Q}_n$.

Construction of RigCol(μ, κ)

Let $\langle \alpha_i : i < \kappa \rangle$ enumerate the singular cardindals of cofinality μ in (μ, κ). Suppose $G_0 \subseteq \mathbb{P}_0$ is generic over V. Let X_0 be the subset of κ that codes G_0. In $V[G_0]$, we define a partial order $\mathbb{P}_1$ and a projection $\pi_1 : \mathbb{Q}_1 \to \mathbb{P}_1$. For $p \in \mathbb{Q}_1$, let

$$\pi_1(p)(\alpha, \beta, \gamma) = \begin{cases} p(\alpha, \beta, \gamma) \text{ if } \alpha = \alpha_i^{++}, \text{ where } i \in X_0, \text{ and} \\ \text{undefined otherwise.} \end{cases}$$

$\mathbb{P}_1$ is simply the range of π_1.

- π_1 is a projection.
- If $i \notin X_0$, then $\Vdash_{\mathbb{P}_1}^{V[G_0]} \neg \operatorname{Spl}(\alpha_i^{++}, \kappa, V)$.
- Whenever $G_0 * G_1$ is $\mathbb{P}_0 * \mathbb{P}_1$-generic over V, and $G_0' * G_1' \in V[G_0 * G_1]$ is also $\mathbb{P}_0 * \mathbb{P}_1$-generic over V, then $G_0 = G_0'$.

Construction of RigCol(μ, κ)

Now we simply continue this process ω many times. Suppose that we have sequences $\langle \mathbb{P}_j : j \leq n \rangle$, $\langle \pi_j : j \leq n \rangle$, and $\langle X_j : j \leq n \rangle$ such that for $1 \leq m \leq n$,

1. $\mathbb{P}_m$ is a subset of $\mathbb{Q}_m$ defined in the extension by $\mathbb{P}_0 * \cdots * \mathbb{P}_{m-1}$.
2. X_m is a $(\mathbb{P}_0 * \cdots * \mathbb{P}_m)$-name for the subset of κ which codes the generic G_m for $\mathbb{P}_m$.
3. It is forced by $\mathbb{P}_0 * \cdots * \mathbb{P}_{m-1}$ that $\pi_m : \mathbb{Q}_m \to \mathbb{P}_m$ is the projection defined by restriction to $\{\alpha_i^{+m+1} : i \in X_{m-1}\} \times \kappa \times \kappa$.

We extend these properties to a sequence of length $n+1$.

The elements of RigCol(μ, κ) are just the elements of $\mathbb{P} \restriction \bigcup_{n<\omega} A_n$, but their ordering is different. We put $p \leq_{\text{RigCol}(\mu,\kappa)} q$ when for each n, $\langle p \restriction A_0, \pi_1(p \restriction A_1), \ldots, \pi_n(p \restriction A_n)\rangle \leq \langle q \restriction A_0, \pi_1(q \restriction A_1), \ldots, \pi_n(q \restriction A_n)\rangle$ in $\mathbb{P}_0 * \cdots * \mathbb{P}_n$.

Projections

Suppose $\mu < \kappa < \delta$ are regular with κ Mahlo. There are projections:

- From $\text{RigCol}(\mu, \delta)$ to $\text{RigCol}(\mu, \kappa)$.
- From $\text{RigCol}(\mu, \delta)$ to $\text{RigCol}(\kappa, \delta)$.
- By an argument of Shioya, from $\text{RigCol}(\mu, \delta)$ to $\text{RigCol}(\mu, \kappa) * \dot{\text{RigCol}}(\kappa, \delta)$.

Saturated rigid ideals

Suppose $j : V \to M$ is an almost-huge embedding with $\text{crit}(j) = \kappa$, $j(\kappa) = \delta$ Mahlo, $\mu < \kappa$ regular.

Let $G * H \subseteq \text{RigCol}(\mu, \kappa) * \dot{\text{RigCol}}(\kappa, \delta)$ be generic. If we force with $\text{RigCol}(\mu, \delta)/(G * H)$, then we can extend the embedding to

$$\hat{j} : V[G * H] \to M[\hat{G} * \hat{H}].$$

In $V[G * H]$ there is a normal ideal I on $\kappa = \mu^+$ such that $\mathcal{P}(\kappa)/I \cong \text{RigCol}(\mu, \delta)/(G * H)$.

$\mathcal{P}(\kappa)/I$ is rigid, because otherwise we would have a $\text{RigCol}(\mu, \delta)$-extension $V[\hat{G}]$ in which there is a generic $\hat{G}' \neq \hat{G}$.

Successors of singulars

Suppose μ is indestructibly supercompact and $\kappa > \mu$ is Mahlo. RigCol(μ, κ) preserves the measurability of μ.

Lemma

Let $X \subseteq \kappa$ be a set of regular cardinals such that for some regular $\nu \in (\mu, \kappa)$, $\nu^+ \notin X$. Let

$$\mathbb{P} = \prod_{\alpha \in X}^{E} \mathrm{Col}(\alpha, {<}\kappa).$$

Let $\mathbb{Q}$ be Prikry forcing at μ after $\mathbb{P}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ forces $\neg\,\mathrm{Spl}(\nu^+, \kappa)$.

Successors of singulars

Theorem (Foreman)

If I is a precipitous ideal on κ and $\mathbb{P}$ is κ-c.c., then

$$\mathbb{P} * \mathcal{P}(\kappa)/\bar{I} \cong \mathcal{P}(\kappa)/I * j(\mathbb{P}).$$

Suppose $\mu < \kappa < \delta$ are as before, with μ indestructible. Let $G * H * K \subseteq \text{RigCol}(\mu, \kappa) * \dot{\text{RigCol}}(\kappa, \delta) * \dot{\mathbb{Q}}$ be generic.

Since $\mathbb{Q}$ is μ-centered, it preserves the saturated ideal on κ. We have

$$\mathcal{P}(\kappa)/\bar{I} \cong (\text{RigCol}(\mu, \delta) * j(\mathbb{Q}))/(G * H * K).$$

A nontrivial automorphism of $\mathcal{P}(\kappa)/\bar{I}$ would give an $\text{RigCol}(\mu, \delta) * j(\mathbb{Q})$-extension $V[\hat{G} * \hat{K}]$ with a different generic $\hat{G}' * \hat{K}'$, with *the same Prikry sequence* associated to $\hat{K}$ and $\hat{K}'$.

As before, $\hat{G} = \hat{G}'$. But then $\hat{K} = \hat{K}'$. Contradiction.

Further results and questions

Using Radin forcing with interleaved collapses, we can get a model of ZFC + GCH where every successor cardinal carries a rigid saturated ideal. This requires some preservation lemmas about the failure of splitting.

We can also get, for any prescribed successor cardinal κ, a model of GCH where $\mathcal{P}(A)/\mathrm{NS}_\kappa$ is rigid and saturated for some stationary $A \subseteq \kappa$.

Questions:

1. Can we get this globally?
2. Are there other applications of RigCol?
3. (Karagila) Suppose κ is inaccessible, and $\mathbb{P}$ is κ-c.c. of size κ forcing $\kappa = \omega_1$, and $\mathbb{P}$ forces unique generics. Is κ Mahlo?

Thank you for your attention!