# On Steinhaus properties and families of "small" sets

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All groups considered here are Abelian.

A family $\mathcal{F}$ of subsets of a Polish group $X$ is defined to be

- *Steinhaus-Smital* if for any Borel subsets $A, B \notin \mathcal{F}$ the difference $A - B$ has non-empty interior in $X$;
- *weak Steinhaus-Smital* if for any Borel subsets $A, B \notin \mathcal{F}$ the difference $A - B$ is non-meager in $X$;
- *Steinhaus* if for any Borel subset $A \notin \mathcal{F}$ the difference $A - A$ is a neighborhood of zero in $X$;
- *weak Steinhaus* if for any Borel subset $A \notin \mathcal{F}$ the difference $A - A$ is non-meager in $X$.

It is clear that

$$\begin{array}{ccc} \text{Steinhaus-Smital} & \Rightarrow & \text{weak Steinhaus-Smital} \\ & & \Downarrow \\ \text{Steinhaus} & \Rightarrow & \text{weak Steinhaus} \end{array}$$

Steinhaus-Smital $\not\Rightarrow$ Steinhaus

## Example 1 (Banakh-Głąb-J.-Swaczyna)

Let $\mathcal{F}$ be the semi-ideal of all subsets $A \subset \mathbb{C}$ such that for any unit circle $S(z_0; 1)$ the intersection $A \cap S(z_0; 1)$ has empty interior in $S(z_0; 1)$. Then $\mathcal{F}$ is Steinhaus-Smital but not Steinhaus in $\mathbb{C}$.

## Problem 1

Is there an ideal $\mathcal{I}$ of subsets of a Polish group such that $\mathcal{I}$ is Steinhaus-Smital but not weak Steinhaus?

$\mathcal{M}$ – the $\sigma$-ideal of meager sets in a topological space $X$.
$\mathcal{N}$ – the $\sigma$-ideal of sets of Haar measure zero in a locally compact topological group $X$.
$\sigma\overline{\mathcal{N}}$ – the $\sigma$-ideal generated by closed sets of Haar measure zero in a locally compact topological group $X$.

## Steinhaus Theorem (1920)

For every locally compact Polish group the ideal $\mathcal{N}$ is Steinhaus-Smital and Steinhaus.

- H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. 1 (1920), 99-104.

## Pettis–Piccard Theorem (1939, 1951)

For every Polish group the ideal $\mathcal{M}$ is Steinhaus-Smital and Steinhaus.

- S. Piccard, *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*, Mem. Univ. Neuchâtel, vol. 13, Secrétariat Univ., Neuchâtel, 1939.
- B.J. Pettis, *Remarks on a theorem of E. J. McShane*, Proc. Amer. Math. Soc. 2 (1951), 166-171.

## Corollary

For every locally compact Polish group the ideal $\mathcal{M} \cap \mathcal{N}$ is Steinhaus.

## Example (Bartoszewicz-Filipczak-Natkaniec, 2011)

For every locally compact Polish group the ideal $\mathcal{M} \cap \mathcal{N}$ is not Steinhaus-Smital.


- A. Bartoszewicz, M. Filipczak, T. Natkaniec, *On Smital properties*, Topology Appl. 158 (2011), 2066-2075.


## Theorem (Banakh-Głąb-J.-Swaczyna)

For every locally compact Polish group the ideal $\sigma\overline{\mathcal{N}} \subset \mathcal{M} \cap \mathcal{N}$ is weak Steinhaus-Smital.

## Example (Banakh-Głąb-J.-Swaczyna)

For the compact metrizable topological group $X = \prod_{n\in\omega} C_{2^n}$, where $C_{2^n} := \mathbb{Z}/2^n\mathbb{Z}$, the ideal $\sigma\overline{\mathcal{N}}$ is neither Steinhaus-Smital nor Steinhaus. ($A$ is a $G_\delta$-subset of the set $\prod_{n\in\omega} C^*_{2^n}$)

For every locally compact Polish group:

| Ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{M} = \sigma\overline{\mathcal{M}}$ | Y | Y | Y | Y |
| $\mathcal{N}$ | Y | Y | Y | Y |
| $\mathcal{M} \cap \mathcal{N}$ | N | Y | Y | Y |
| $\sigma\overline{\mathcal{N}}$ | N | N | Y | Y |

## Definition 1 (Christensen, 1972)

A subset $A$ of a Polish group $X$ is called *Haar null*, if there are a Borel set $B \subset X$ containing $A$ and a probability $\sigma$-additive Borel measure $\mu$ on $X$ such that $\mu(x + B) = 0$ for each $x \in X$.

- The family $\mathcal{HN}$ of all Haar null sets is a $\sigma$-ideal.
- If $X$ is locally compact, then $\mathcal{HN} = \mathcal{N}$.

## Theorem (Christensen, 1972, Matoušková-Zelený, 2003)

In every Polish group the ideal $\mathcal{HN}$ is Steinhaus.
In any non-locally compact Polish group $\mathcal{HN}$ is not Steinhaus-Smital.


- J.P.R. Christensen, *On sets of Haar measure zero in abelian Polish groups*, Israel J. Math. 13 (1972), 255-260.
- E. Matoušková, M. Zelený, *A note on intersections of non–Haar null sets*, Colloq. Math. 96 (2003), 1-4.

## Definition 2 (Darji, 2013)

Let $X$ be a Polish group. A set $A \subset X$ is called *Haar meager*, if there exists a Borel set $B \subset X$ containing $A$, a compact metric space $K$ and a continuous function $f : K \to X$ such that $f^{-1}(B + x)$ is meager in $K$ for each $x \in X$.

- The family $\mathcal{HM}$ of all Haar meager sets is a $\sigma$-ideal.
- If $X$ is locally compact, then $\mathcal{HM} = \mathcal{M}$.
- If $X$ is not locally compact, then $\mathcal{HM} \subsetneq \mathcal{M}$.

- U.B. Darji, *On Haar meager sets*, Topology Appl. 160 (2013), 2396-2400.

## Theorem (J., 2015)

In every Polish group the ideal $\mathcal{HM}$ is Steinhaus.
In any non-locally compact Polish group $\mathcal{HM}$ is not Steinhaus-Smital.

- E. Jabłońska, *Some analogies between Haar meager sets and Haar null sets in abelian Polish groups*, J. Math. Anal. Appl. 421 (2015), 1479-1486.

## Example (folklore)

For the group $X = \mathbb{Z}^\omega$ the ideals $\mathcal{HN}$ and $\mathcal{HM}$ are not weak Steinhaus-Smital. ($A = \omega^\omega$)

In a locally compact Polish group:

| Ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{M} = \mathcal{HM}$ | Y | Y | Y | Y |
| $\mathcal{N} = \mathcal{HN}$ | Y | Y | Y | Y |

In a non-locally compact Polish group:

| Ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{M}$ | Y | Y | Y | Y |
| $\mathcal{HM}$ | N | Y | N | Y |
| $\mathcal{HN}$ | N | Y | N | Y |

**Theorem (Banakh-Głąb-J.-Swaczyna)**

Each closed Haar null subset of a Polish group is Haar meager.
Consequently $\overline{\mathcal{HN}} \subset \overline{\mathcal{HM}}$ and $\sigma\overline{\mathcal{HN}} \subset \sigma\overline{\mathcal{HM}}$.

**Example 2 (Banakh-Głąb-J.-Swaczyna)**

The Polish group $\mathbb{Z}^\omega$ contains a meager $\sigma$-Polish subgroup $Y$, which does not belong to the ideal $\sigma\overline{\mathcal{HM}}$ in $\mathbb{Z}^\omega$, Consequently, $\sigma\overline{\mathcal{HM}}$ is not weak Steinhaus.

In a locally compact Polish group:

| Ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\sigma\overline{\mathcal{HN}} = \sigma\overline{\mathcal{N}}$ | N | N | Y | Y |
| $\sigma\overline{\mathcal{HM}} = \sigma\overline{\mathcal{M}} = \mathcal{M}$ | Y | Y | Y | Y |

In a non-locally compact Polish group:

| Ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\sigma\overline{\mathcal{HN}}$ | N | N | N | N |
| $\sigma\overline{\mathcal{HM}}$ | N | N | N | N |

## Definition 3 (Dodos, 2004)

Let $\mathcal{P}(X)$ be the space of all probability $\sigma$-additive Borel measures on a Polish group $X$. For a Borel set $A \subset X$

$$T(A) := \{\mu \in \mathcal{P}(X) : \ \mu(x + A) = 0 \ \text{ for all } \ x \in X\}.$$

## Theorem (Dodos, 2004)

If $A$ is a Borel Haar null subset of a Polish group $X$, then $T(A)$ is either meager or comeager in $\mathcal{P}(X)$.


- P. Dodos, *Dichotomies of the set of test measures of a Haar null set*, Israel J. Math. 144 (2004), 15-28.
- P. Dodos, *On certain regularity properties of Haar-null sets*, Fund. Math. 181 (2004), 97-109.

## Definition 4 (Dodos, 2004)

A subset $A$ of a Polish group $X$ is called *generically Haar null* provided $T(A)$ is comeager in $\mathcal{P}(X)$.

$\mathcal{GHN}$ – the $\sigma$-ideal generated by Borel generically Haar null sets in $X$.

## Theorem (Dodos, 2009)

The ideal $\mathcal{GHN}$ in a Polish group $X$ is weak Steinhaus.


- P. Dodos, *The Steinhaus property and Haar-null sets*, Bull. Lond. Math. Soc. 41 (2009), 377–384.


## Proposition (Banakh-Głąb-J.-Swaczyna)

Let $A \subset X$ be a Borel set. The following conditions are equivalent:

(i) $A$ is Haar meager;

(ii) there exists a continuous function $f : 2^\omega \to X$ such that the set $f^{-1}(A + x)$ is meager in $2^\omega$ for $x \in X$.

### Definition 5 (Banakh-Głąb-J.-Swaczyna)

Let $C(2^\omega, X)$ be the space of all continuous functions $f : 2^\omega \to X$, where $X$ is a Polish group. For a Borel set $A \subset X$

$$W(A) := \{f \in C(2^\omega, X) : \ f^{-1}(x + A) \ \text{is meager in} \ 2^\omega \ \text{for all} \ x \in X\}.$$

### Theorem (Banakh-Głąb-J.-Swaczyna)

If $A$ is a Borel Haar meager subset of a Polish group $X$, then $W(A)$ is either meager or comeager in $C(2^\omega, X)$.

### Definition 6 (Banakh-Głąb-J.-Swaczyna)

A subset $A$ of a Polish group $X$ is called *generically Haar meager* provided $W(A)$ is comeager in $C(2^\omega, X)$.

$\mathcal{GHM}$ – the $\sigma$-ideal generated by Borel generically Haar meager sets in $X$.

**Theorem (Banakh-Karchevska-Ravsky, 2015)**

The ideal $\mathcal{GHM}$ in a Polish group $X$ is weak Steinhaus.


- T. Banakh, L. Karchevska, A. Ravsky, *The closed Steinhaus properties of $\sigma$-ideals on topological groups*, arXiv:1509.09073v1 [math.GN] 30 Sep 2015.

In a locally compact Polish group:

$$\sigma\overline{\mathcal{N}} \subset \mathcal{GHN} \subset \mathcal{M} \cap \mathcal{N}$$

| Semi-ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{GHM} = \mathcal{M}$ | Y | Y | Y | Y |
| $\mathcal{M} \cap \mathcal{N}$ | N | Y | Y | Y |
| $\mathcal{GHN}$ | N | ? | Y | Y |
| $\sigma\overline{\mathcal{N}}$ | N | N | Y | Y |

In a non-locally compact Polish group:

$$\mathcal{GHN} \subset \mathcal{HN}, \quad \mathcal{GHM} \subset \mathcal{HM}$$

| Semi-ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{HM}$ | N | Y | N | Y |
| $\mathcal{HN}$ | N | Y | N | Y |
| $\mathcal{GHM}$ | N | ? | N | Y |
| $\mathcal{GHN}$ | N | ? | N | Y |

In every **locally compact** Polish group:

$$\sigma\overline{\mathcal{N}} = \sigma\overline{\mathcal{HN}} \subset \mathcal{GHN} \subset \mathcal{M} \cap \mathcal{N} \subset \mathcal{HN} = \mathcal{N},$$

$$\sigma\overline{\mathcal{M}} = \sigma\overline{\mathcal{HM}} = \mathcal{GHM} = \mathcal{HM} = \mathcal{M}.$$

| Semi-ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{M} = \mathcal{HM} = \mathcal{GHM} = \sigma\overline{\mathcal{M}}$ | Y | Y | Y | Y |
| $\mathcal{N} = \mathcal{HN}$ | Y | Y | Y | Y |
| $\mathcal{M} \cap \mathcal{N}$ | N | Y | Y | Y |
| $\mathcal{GHN}$ | N | ? | Y | Y |
| $\sigma\overline{\mathcal{N}}$ | N | N | Y | Y |

In every **non-locally compact** Polish group:

$$\sigma\overline{\mathcal{HN}} \subset \mathcal{GHN} \subset \mathcal{M} \cap \mathcal{HN} \subset \mathcal{HN},$$

$$\sigma\overline{\mathcal{HM}} \subset \mathcal{GHM} \subset \mathcal{HM} \subset \mathcal{M}.$$

| Semi-ideal | S-S | S | weak S-S | weak S |
|---|---|---|---|---|
| $\mathcal{M} = \sigma\overline{\mathcal{M}}$ | Y | Y | Y | Y |
| $\mathcal{HN}$ | N | Y | N | Y |
| $\mathcal{HM}$ | N | Y | N | Y |
| $\mathcal{GHM}$ | N | ? | N | Y |
| $\mathcal{GHN}$ | N | ? | N | Y |
| $\sigma\overline{\mathcal{HM}}$ | N | N | N | N |
| $\sigma\overline{\mathcal{HN}}$ | N | N | N | N |

| Semi-ideal | S-S | | weak S-S | | S | | weak S | |
|---|---|---|---|---|---|---|---|---|
| | l.c. | n.l.c. | l.c. | n.l.c. | l.c. | n.l.c. | l.c. | n.l.c. |
| $\mathcal{M}$ | Y | Y | Y | Y | Y | Y | Y | Y |
| $\mathcal{HN}$ | Y | N | Y | Y | Y | N | Y | Y |
| $\mathcal{HM}$ | Y | N | Y | Y | Y | N | Y | Y |
| $\mathcal{GHM}$ | Y | N | Y | ? | Y | N | Y | Y |
| $\mathcal{GHN}$ | N | N | ? | ? | Y | N | Y | Y |
| $\sigma\overline{\mathcal{HN}}$ | N | N | N | N | Y | N | Y | N |
| $\sigma\overline{\mathcal{HM}}$ | Y | N | Y | N | Y | N | Y | N |