Truth Theories and Satisfaction Classes over Models of Set Theory

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Satisfaction Classes

The following can be done in any first-order theory T in a recursive language $\mathcal{L}$ strong enough to manipulate syntax (interpreting PA or even $I\Delta_0 + exp$ is enough).
Fix arithmetical formulae: $Trm_{\mathcal{L}}(t)$ that expresses *t is a term*, a formula $Frm_{\mathcal{L}}(\varphi)$ that expresses *φ is a formula*, and a formula $VAsn(a, \varphi)$ that expresses *a is a variable assignment for φ*. Then $val(t, a)$ denotes the evaluation of t under a.

Definition (Satisfaction Class)

Let $\mathcal{M} \models T$ and let $F \subseteq Frm^{\mathcal{M}}_{\mathcal{L}}$.
A subset $S \subseteq F \times M$ is an F-**satisfaction class** if the following conditions are satisfied:

1. if $(\varphi, a) \in S$, then $Vasn(a, \varphi)$ (and $Frm_{\mathcal{L}}(\varphi)$).
2. if $Trm_{\mathcal{L}}(t_k)$ for $k \leq n$ and $\varphi = R(t_1, \ldots, t_k)$, then $(\varphi, a) \in S$ iff $VAsn(a, R(t_1, \ldots, t_k))$ and $R(val(t_1, a), val(t_2, a), \ldots, val(t_n, a))$.
3. for all $\varphi, \psi \in F$ and for all $a \in M$: $(\varphi \vee \psi, a) \in S$ iff $Vasn(a, \varphi \vee \psi)$ and $\exists a' \subseteq a\, (\varphi, a') \in S$ or $(\psi, a') \in S$.
4. for all $\varphi \in F$ and for all $a \in M$: $(\neg\varphi, a) \in S$ iff $Vasn(a, \neg\varphi)$ and $(\varphi, a) \notin S$.
5. for all $\exists x_i \varphi(x_i) \in F$ and for all $a \in M$: $(\exists x_i \varphi(x_i), a) \in S$ iff $(\varphi, a^\frown(i, m)) \in S$ for some $m \in M$.

Satisfaction Classes and Tarski's Theorem

The definition of a satisfaction class is a single sentence in $\mathcal{L}$ with an extra predicate symbol S - call it $Sat(S)$. If $F = Frm_{\mathcal{L}}^{\mathcal{M}}$, then the satisfaction class is **full**. Unless otherwise stated, all satisfaction classes below are full.

Tarski's Theorem gives the following (and much more, actually):

if S is a full satisfaction class and $(\mathcal{M}, S) \models Sat(S)$, then S is undefinable in $\mathcal{M}$.

Compositional Truth

Definition (Typed Compositional Truth Theory)

Let T be as above. Typed Compositional Truth Theory ($CT^-[T]$) is an axiomatic theory obtained from T by adjoining to it the following axioms:

1. $\forall s, t \in Trm[Tr(s = t) \equiv val(s) = val(t)]$,
2. $\forall x[Sent_{\mathcal{L}}(x) \Rightarrow (Tr(\neg x) \equiv \neg Tr(x))]$,
3. $\forall x, y[(Sent_{\mathcal{L}}(x \wedge y)) \Rightarrow (Tr(x \wedge y) \equiv Tr(x) \wedge Tr(y))]$,
4. $\forall x, y[(Sent_{\mathcal{L}}(x \vee y)) \Rightarrow (Tr(x \vee y) \equiv Tr(x) \vee Tr(y))]$,
5. $\forall v, x[Sent_{\mathcal{L}}(\forall vx) \Rightarrow (Tr(\forall vx) \equiv \forall n Tr(x(n/v)))]$,
6. $\forall v, x[Sent_{\mathcal{L}}(\exists vx) \Rightarrow (Tr(\exists vx) \equiv \exists n Tr(x(n/v)))]$,

where Tr is a fresh unary predicate and where by $Sent_{\mathcal{L}}(x)$ we mean that x is the Gödel number af a sentence of the recursive language $\mathcal{L}$ (i.e. Tr applies only to sentences without any occurence of the truth predicate).

Truth Theories and Satisfaction Classes

Definition

Let a, b be assignments. Then $(\varphi, a) \approx (\psi, b)$ iff $sub(\varphi, a) = sub(\psi, b)$.
S is an extensional full satisfaction class in a model $\mathcal{M}$ iff S is a satisfaction class and

$$(M, S) \models \forall \varphi, \psi \, \forall a, b \ [(\varphi, a) \approx (\psi, b) \Rightarrow ((\varphi, a) \in S \equiv (\psi, b) \in S)].$$

Fact

Let $\mathcal{M} \models T$. The following are equivalent:

1. *There is an extensional satisfaction class in $\mathcal{M}$,*
2. *$\mathcal{M}$ can be expanded to a model of $CT^{-}[T]$.*

Proof.

Given S, define:

$$Tr^{\mathcal{M}} := \{\varphi \in Sent_{\mathcal{L}}^{\mathcal{M}} : (\mathcal{M}, S) \models \exists a \, (\varphi, a) \in S)\}.$$

Given Tr, define:

$$S^{\mathcal{M}} := \{(\varphi, a) \in Frm_{\mathcal{L}}^{\mathcal{M}} \times M : \ (\mathcal{M}, Tr) \models Tr(sub(\varphi, a))\}.$$

□

Types

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Definition

- *A set p of the formulae of the language $\mathcal{L}_M$ (i.e. the language $\mathcal{L}$ extended with a constant name for every element of the model $\mathcal{M}$) with exactly one free variable x is **finitely satisfied** in $\mathcal{M}$ if and only if for any finite $q \subset p$ there exists an $a \in M$ such that for any $\varphi(x) \in q$ $\mathcal{M} \models \varphi(a)$.*
- *A **type** over a model $\mathcal{M}$ is a finitely satisfied set of formulae of the form $\varphi(x, b)$ with exactly one free variable x and at most one parameter $b \in M$.*
- *A type p over $\mathcal{M}$ is **recursive** if and only if the set of codes of formulae $\varphi(x, y)$ such that $\varphi(x, b) \in p$ is recursive.*
- *A type p over $\mathcal{M}$ is (globally) **realised** if and only if there exists an $a \in M$ such that for any $\varphi(x, b) \in p$ $\mathcal{M} \models \varphi(a, b)$.*
- *A model $\mathcal{M}$ is **recursively saturated** if and only if each recursive type over $\mathcal{M}$ is realised.*

Usefulness of truth via recursive saturation

What's the use of recursively saturated structures?
They e.g. have nice algebraic and combinatorial properties, e.g.

Every countable recursively saturated first-order structure is **strongly ω-homogeneous**, which means that if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ finite tuples from the structure and they realize the same set of formulae, then there is an automorphism f such that $f(a_1, \ldots, a_n) = b_1, \ldots, b_n$.

Thus: recursively saturated models have rich automorphism groups.

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1. Recusrisvely saturated models are (chronically) resplendent (i.e. they satisfy as many Σ^1_1 properties as possible, as well as their expansions).
2. By Skolem-Lowenheim and Elementary Chain theorems, we can often restrict one's attention to them while studying models of a given FO-theory for general conclusions, i.e. every infinite model $\mathcal{M}$ has a recursively saturated elementary extension of the same cardinality.

Recursive Saturation and models of $CT^-[PA]$

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Theorem (A. Lachlan 1981)

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Theorem (H. Kotlarski and S. Krajewski and A. Lachlan 1981, J. Barwise and J. Schlipf 1976, A. Enayat, and A. Visser 2013)

If a countable model $\mathcal{M} \models PA$ is recursively saturated, than it admits a full satisfaction class.

Satisfaction Classes over models of *ZFC*

Can you have a similar (or maybe the same?) characterisations for models of set theory insted of arithmetic?

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The Enayat - Visser method from 2013 gives us:

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If a countable model $\mathcal{M} \models ZFC$ is recursively saturated, than it admits a full satisfaction class.

Resplendency

Definition (A resplendent model)

*A model M is **resplendent**, if for all tuples $\bar{a} \in M$ of finite length and for all Σ^1_1-sentences $\exists X \Theta(X, \bar{a})$ if $Th(M, \bar{a}) + \exists X \Theta(X, \bar{a})$ is consistent, then*

$$(M, \bar{a}) \models \exists X \Theta(X, \bar{a}).$$

Fact

Every countable recursively saturated model for a recursive language is resplendent.

The Enayat - Visser Theorem for models of set theory

Theorem

If a countable model $\mathcal{M} \models ZFC$ is recursively saturated, than it admits a full satisfaction class.

Lemma

Fix $\mathcal{M}$. Let $F_0 \subseteq Frm_{\mathcal{L}}^{\mathcal{M}}$ that is closed under taking direct subformulae and S_0 an F_0-satisfaction class. Then there is $\mathcal{K} \succeq \mathcal{M}$ with $Frm_{\mathcal{L}}^{\mathcal{M}}$-satisfaction class $S \supseteq S_0$.

Proof of the theorem.

- We can always find a satisfaction class if we move to an elementary extension.
- There are new formulae in the extension, but we can apply the lemma again.
- After ω-many steps, we obtain a model with a full satisfaction class.
- Notice: there are only finitely many Tarski clauses.
- By reseplendency of the starting model we are done.

□

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Proof.

- We let $\mathcal{L}^* = \mathcal{L} \cup \{P_\varphi : \varphi \in Frm_{\mathcal{L}}^{\mathcal{M}}\}$, where each P_φ is a fresh unary predicate.
- $S := \{(\varphi, a) : \varphi \in Frm_{\mathcal{L}}^{\mathcal{M}}$ and $a \in P_\varphi\}$, so that each P_φ is the set of all satisfying assignments for φ, so each P_φ has to satisfy the corresponding Tarski clause τ_φ.
- By compactness, we show the consistency of the following $\mathcal{L}^*(M)$-theory:

$$U := ElDiag(\mathcal{M}) \cup \{\tau_\varphi : \varphi \in Frm_{\mathcal{L}}^{\mathcal{M}}\} \cup$$
$$\cup \{P_\varphi(a) : (\varphi, a) \in S_0\} \cup \{\neg P_\varphi(a) : (\neg\varphi, a) \in S_0\}.$$

- Every finite $U_0 \subseteq U$ is consistent by recursion along the rank given by the relation of being a direct subformula.

□

Lachlan's Theorem for models of set theory

In the standard proof of Lachlan's theorem for models of PA, we rely on the fact that we have full induction (i.e. least number principle) in the universe.
Thus, using the standard proof we may have:

Theorem

If an ω-nonstandard model $\mathcal{M} \models ZFC + V = HOD$ admits a full satisfaction class, then $\mathcal{M}$ is recursively saturated.

Can we drop the well-ordering assumption?

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Yes, we can.

The idea how to do it comes from Bartosz Wcislo's proof of the result that in each nonstandard model $(M, T) \models CT^{-}[PA]$ a partial inductive satisfaction class is definable.

What we need for the proof is the existence of sequences of ω-nonstandard length of certain special formulae.

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Theorem

If an ω-nonstandard model $\mathcal{M} \models ZFC$ admits a full satisfaction class, then $\mathcal{M}$ is recursively saturated.

Definition

Let $(\mathcal{M}, Tr) \models CT^-[ZFC]$ and let $p(x) = \{\varphi_i(x)\ i \in \omega\}$ be any countable type (with parameters) and let $\psi \in Frm_{\mathcal{L}}^{\mathcal{M}}$. The p-rank of ψ ($rk_p(\psi)$) is $n \in \omega$ if n is the least natural number such that:

$$(\mathcal{M}, Tr) \not\models \forall x (\bigwedge_{i=0}^{n} Tr(\psi(x)) \rightarrow \varphi_i(x)) \wedge \exists y\ Tr(\psi(y)).$$

If the $rk_p(\psi)$ is not equal to any $n \in \omega$, then it is ∞.

It is then enough to show for every recursive type p with parametres that there exists a formula φ with $rk_p(\varphi) = \infty$.

Lachlan's Theorem for models of set theory

Theorem

If an ω-nonstandard model $\mathcal{M} \models ZFC$ admits a full satisfaction class, then $\mathcal{M}$ is recursively saturated.

We can obtain the goal with two lemmas:

Lemma

Let $(\mathcal{M}, Tr) \models CT^{-}[ZFC]$ be an ω-nonstandard model and let $d \in \omega^{\mathcal{M}}$ be a nonstandard element. Suppose that $r : \omega^{\mathcal{M}} \cap [0, d] \to \omega+1$ is a function such that for any $a < d$ either $r(a) = \infty$ or $r(a) < r(a+1)$. Then there exists some $b \in \omega^{\mathcal{M}}$ with $r(b) = \infty$

Lemma

Let $(\mathcal{M}, Tr) \models CT^{-}[ZFC]$ be an ω-nonstandard model, let $c > \omega$ and let $p(x) = \{\varphi_n(x) : n \in \omega\}$ be a recursive type of $\mathcal{L}_{\in}$-formulae with parameters. Then there exists a coded sequence $(\psi_i)_{i=0}^{c}$such that for an arbitrary $a < c$, if $rk_p(\psi_a) \neq \infty$, then $rk_p(\psi_a) < rk_p(\psi_{a+1})$.

Conclusion

Corollary

Let $\mathcal{M} \models ZFC$ be a countable ω-nonstandard model. Then the following are equivalent:

1. *$\mathcal{M}$ admits an expansion to a model $(\mathcal{M}, Tr) \models CT^{-}[ZFC]$.*
2. *$\mathcal{M}$ is recursively saturated.*

Therefore:

Corollary

A countable ω-nonstandard model of ZFC admits a compositional truth predicate if and only if it belongs to the so-called natural model of the Multiverse Axioms.

Questions

Can you have a potentialist framework (e.g. modal logic) for models of *ZFC* (or of *PA*) admitting (partial or full, inductive or not) satisfaction classes and statements about them and about the satisfaction classes?
Example: (Enayat-Visser - unpublished): any model of *PA* with a pratial inductive satisfaction class S_0 has an end extension with a full satisfaction class $S_1 \supseteq S_0$.

What about satisfaction classes for infinitary languages or about amenable satisfaction classes?

What about various conservativeness properties (of other axiomatic truth theories)?

Thank you